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Strings in gravimagnetic fields

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Abstract: We provide a complete solution of closed strings propagating in Nappi-Witten space. Based on the analysis of geodesics we construct the coherent wavefunctions which approximate as closely as possible the classical trajectories. We then present a new free field realization of the current algebra using the γ, β ghost system. Finally we construct the quantum vertex operators, for the tachyon, by representing the wavefunctions in terms of the free fields. This allows us to compute the three- and four-point amplitudes, and propose the general result for N-point tachyon scattering amplitude.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, Conformal Field Models in String Theory, Penrose limit and pp-wave background.

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Contents

1. Introduction

Understanding string dynamics in previously inaccessible supergravity backgrounds has been made possible by the realization that on the maximally symmetric backgrounds of PP-waves string theory becomes exactly soluble, even in the presence of Ramond-Ramond fields [1]–[3]. Also it has been known that certain backgrounds with constant field strengths of Neveu-Schwarz B-field are soluble in the same PP-wave limit [4].

There has been a flurry of studies of strings in pp-wave backgrounds prompted by the realization that such models become free in the light-cone gauge. This approach has been successful in understanding the spectrum of the theory. However the light-cone formalism prevents us from dealing with string interactions using familiar worldsheet techniques for computing scattering amplitudes, forcing upon us the light-cone string field theory approach [5]. The main purpose of this paper is to describe string interactions in a pp-wave background in a fully covariant formalism where the worldsheet theory is interacting. We shall only address the question in the context of a pp-wave background with NS-NS flux but no R-R flux. We should mention that this issue has also been taken up recently by G. D'Appollonio and E. Kiritsis [6]. However the approach in our paper is completely different and it allows us to compute any N-point scattering amplitudes, including but not limited to three- and four-points.

The specific model which we consider is the Nappi-Witten model [7]. It does as closely as possible represent the situation of flat space with NS flux, by having metric deformed away from flat space to take into account the curvature introduced by the flux. There is unfortunately no limit available where one may be able to neglect the curvature, and consider the torsion field alone. Thus in order to understand the effect of torsion it will be necessary to disentangle the curvature from torsion *a posteriori*. Certainly it is true that the shape of closed strings is distorted by the curvature and torsion. For large values of the torsion the usual vacua are unstable and new vacua, the “long-string states' of [8] carrying winding numbers are created. We discuss this dynamical dielectric-type effect in section 2.2.

Though formally similar to the SL$(2,\mathbb{R})$ and the SU(2) WZW models, the Nappi-Witten model is in many ways simpler. For example the naive value $c = 4$ for the central charge holds [7], it is equal to the dimension of the group. Considerable simplification is due to the particularly simple form of the cubic interactions on the worldsheet as well as the fact that Nappi-Witten background being an exact solution of the string sigma model. The main technical reason which allows us to completely solve the theory in this case is the fact that we have found a mapping of the Nappi-Witten model to a free field theory. We can then identify the symmetry currents and construct the vertex operators in terms of the free fields.

We should emphasize that our approach is very different from the earlier attempts [9, 10, 11] to solve the NW model in terms of quasi-free (or twist) fields which do not allow an explicit construction of the vertex operators. Our proposal for the N-point tachyon amplitude is new. So is a relation between amplitudes having a different number of conjugate fields. Our results for the three and four points function agree with the ones in [6].

We have tried in this paper to give a complete and consistent description of closed strings propagating in Nappi-Witten space. We review in section 2 the classical picture of string propagation in the light-cone gauge and in section 2.3 the semi-classical wave propagation. The classical analysis gives us a nice physical picture and provides us with the geometric intuition to guide us in the covariant (algebraic) approach.

The Wakimoto free field realization of the algebra, using β, γ "ghosts" is introduced in section 3. This is formally similar to the one used for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$. We are able to give a geometrical interpretation of these subsidiary fields in terms of the original coordinate fields not only for γ, which is a simple coordinate redefinition, but also for the β field. We then construct the vertex operators in terms of these fields and the conjugate vertex operators via an integral transformation. These constructions are presented in section 4. The extra simplification, as compared to $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ cases, is due to the fact that the screening charge contains only null fields and thus producing no contractions when we perform the vertex operator correlator calculation (in section 5). The final expression for the N-point amplitude with one conjugate vertex operator is, instead of a complicated multiple integral of the Fateev-Dotsenko type, a single surface integral in just one complex variable. We present some checks on this amplitude in section 6 using conformal and algebraic Ward identities as well as the Knizhnik-Zamolodchikov equation. The N-point amplitude with more than one conjugate field is given in term of an integral transform of the original amplitude. We also present explicit justifications for this rule in the case of three and four point amplitudes. The "long strings" appear already as poles in the three-point amplitudes. The four-point amplitudes and the factorization property are analyzed. Finally we discuss the flat space limit of our general N-point amplitudes in section 6.4.

Several appendices contain detailed derivations which serve to make our paper self-contained as well as to correct some misprints in the literature. Appendix A contains a thorough analysis in the light cone gauge. Appendix B reviews the representation theory of the Nappi-Witten algebra and its link with the wave functionals on the group manifold. In appendix C we derive the integral transform of the conjugate vertex operators. Finally appendix D details proof of the chiral splitting formulae used in the computation of the correlation functions.

2. Strings on Nappi-Witten spacetime

The Nappi-Witten model [12] is a WZW model on the centrally-extended two-dimensional Poincare group. The solvability of string theory in this group manifold(the Nappi-Witten space) relies on this underlying infinite dimensional symmetry.[1] $J_{\pm}, J, T$ are the anti-

[1] The existence of infinite dimensional symmetry may also play a role in solving string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ [13, 14].

hermitean generators of the algebra:

$$[J^+,\, J^-] = 2\,i\,T\,, \qquad [J,\, J^+] = i\,J^+\,, \qquad [J,\, J^-] = -i\,J^-\,, \qquad [T,\, \diamondsuit] = 0\,. \tag{2.1}$$

A generic group element is given by

$$g(a,u,v) = e^{HaJ^+ + H\bar{a}J^-}\, e^{HuJ + HvT}\,, \tag{2.2}$$

where $a = \frac{1}{2}(a^1 + i\,a^2)$. The group properties are spelled out in appendix B.

The Nappi-Witten space is the group manifold, parametrized by $a, \bar{a}, u, v$, whose metric is given by

$$\begin{aligned} ds^2 &= \frac{1}{H^2}\mathrm{tr}(g^{-1}\,dg)^2 \\ &= 2\,du\,dv + 4da\,d\bar{a} + 2i\,(\bar{a}\,da - a\,d\bar{a})\,du\,. \end{aligned} \tag{2.3}$$

In addition there is a NS B-field with constant field strength H:

$$B_{ui} = -H\,\epsilon_{ij}\,a^j \qquad H_{u12} = -H\,. \tag{2.4}$$

A simple analysis shows that the coordinate direction corresponding to J is a null direction which should be identified with the $u = t + \psi$ of the pp-wave limit. The fact that translations along the light cone time do not commute with spatial translations but instead generate rotations in the spatial plane should be interpreted as angular momentum carried by the circularly polarized plane wave.

The metric eq. (2.3) is invariant under the isometry group $G_L \times G_R$, $g \to g_L^{-1}\,g\,g_R$. Infinitesimally the group action is given by

$$\begin{aligned} T_L &= -\partial_v\,, & T_R &= \partial_v\,, \\ J_L &= -(\partial_u + i\,(a\,\partial_a - \bar{a}\,\partial_{\bar{a}}))\,, & J_R &= \partial_u\,, \\ J_L^+ &= -(\partial_a + iH\bar{a}\,\partial_v)\,, & J_R^+ &= e^{iHu}(\partial_a - iH\bar{a}\,\partial_v)\,, \\ J_L^- &= -(\partial_{\bar{a}} - iHa\,\partial_v)\,, & J_R^- &= e^{-iHu}(\partial_{\bar{a}} + iHa\,\partial_v)\,. \end{aligned} \tag{2.5}$$

The generator T generates translations in v, J_R generates translations in the u direction, and $J_L + J_R$ rotations in the transverse plane. The others generate some twisted translations in the transverse plane. Overall, the isometry group is 7-dimensional[2] and consists of two commuting copies of the Nappi-Witten algebra. Remarkably boosts are not among these symmetries even in the limit of flat space $H \to 0$. This makes it impossible to use old techniques of setting the light-cone momenta to zero and later using the symmetries to recover the fully general results.

Some remarks are due on this nonabelian group of isometry. First it is impossible to diagonalize all the generators at once. Second, the mutually commuting but inequivalent action of the descendants of the other chiral copy is to be taken into account. The net result is that a state is characterized by four quantum numbers: two real momenta p^+, p^- and two complex numbers ρ and λ which parametrize the position and the radius of the

[2] The commuting generator T should be counted only once.

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corresponding classical trajectory. One may be surprised that the position is a quantum number characterizing a state. But as we shall see, λ and ρ are indeed the eigenvalues of some of the isometry generators: they are the zero mode part of the current algebra.

Finally the corresponding string sigma model action reads[3]

$$S(g) = \frac{1}{4\pi}\int d\sigma^2\Big\{\sqrt{-h}\,h^{\alpha\beta}\,[\,2\,\partial_\alpha u\,\partial_\beta v + \partial_\alpha a^i\,\partial_\beta a^i - H\,(a^1\,\partial_\alpha a^2 - a^2\,\partial_\alpha a^1)\,\partial_\beta u\,] - \\ - H\,\epsilon^{\alpha\beta}\,(a^1\,\partial_\alpha a^2 - a^2\,\partial_\alpha a^1)\,\partial_\beta\,u\Big\}\,, \tag{2.6}$$

where H is related to the level k of the WZW model by $k = H^{-2}$. After conformal gauge-fixing the terms due to curvature and torsion can be naturally combined:

$$S(g) = \frac{1}{4\pi}\int 2\,\partial_+\,u\partial_- v + \partial_+ a^i\,\partial_- a^i - H\,(a^1\,\partial_- a^2 - a^2\,\partial_- a^1)\,\partial_+ u\,. \tag{2.7}$$

And hence our observation that the effects of curvature and torsion are inseparable, in the sense that there is no (obvious) way to take a limit in which the metric perturbation would go away while retaining the torsion. Note that if we rescale u by H and v by H^{-1} we can absorb the dependance in the coupling constant. We shall therefore suppose for most parts of the paper that $H = 1$. We shall reinstate the H dependence at the very end when we discuss the flat space limits of our N-point amplitude.

2.1 Geodesic motion

A lot of intuition about the behaviour of strings can be gained by first looking at the behaviour of their centre-of-mass coordinates, i.e. the classical picture of particles propagating in the given geometry.

The geodesic, followed from the metric (eq.(2.3)), is described by:

$$\begin{aligned} u &= u_0 + p^+\,\tau \\ a &= -\rho + \lambda\,e^{+ip^+\tau}\,, \\ \bar{a} &= -\bar{\rho} + \bar{\lambda}\,e^{-ip^+\tau}\,. \end{aligned} \tag{2.8}$$

As advertised above, the integration constants ρ, λ represent the position and radius of the circular trajectory. Having found $a, \bar{a}$ we can now solve the remaining equation for v

$$v = v_0 + (p^- + 2\,p^+\,|\lambda|^2)\,\tau + i\left(\bar{\lambda}\,\rho\,e^{-ip^+\tau} - \lambda\,\bar{\rho}\,e^{+ip^+\tau}\right). \tag{2.9}$$

Lastly, if we are interested in massless particles, the condition for the world-line to be light-like is

$$2\,p^+\,p^- + |2p^+\lambda|^2 = 0\,.$$

As seen from this identity we can interpret $2p^+\lambda$ to be the momentum of transverse coordinates, $p_\perp$. This gives the parameter λ a dual role: on the one hand it is the radius of the transverse circle, on the other hand it is the transverse momentum in units of p^+. We shall see later in subsection 2.2 that at the quantum level this fact will manifest itself as noncommutativity in the transverse space.

[3] We are ignoring the other six directions. We have also omitted the fermions.

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Closed strings having sufficiently small light cone momenta do follow geodesics as expected on general grounds. The reason is that in deriving the geodesic equation from string equation of motion one assumes the ground state to be σ-independent. This is consistent with the closed string periodic boundary conditions. However if the light cone momentum becomes big, the σ-independent configuration is not necessarily a minimum energy state. New vacua corresponding to long-string configurations, which have nontrivial σ dependence, emerge. So the effect of having a torsion field extending in the time direction leads to dynamical effects under which the geodesic is no longer the natural motion of the closed string.

For open strings the torsion has even more drastic effects. It affects the centre-of-mass motion of the open string and hence its motion deviates from geodesics. The two ends of the open string couple directly to $B + F$ with opposite signs and the string becomes polarized as a dipole. When $B + F$ is non-uniform the open string will experience a net force proportional to the gradient of the field. The situation is furthermore complicated by the fact that the resulting physical description depends on the electromagnetic field, F, in the background.

2.2 Light cone analysis

We would now go to the light-cone gauge in which the cubic interaction term becomes quadratic and the model becomes soluble. The light-cone hamiltonian, $\mathcal{H}_{lc}$, is

$$\mathcal{H}_{lc} = \frac{1}{2}\Pi_1^2 + \frac{1}{2}\Pi_2^2 + \mu\,(a_1\Pi_2 - a_2\Pi_1) + \frac{1}{2}\acute{a}_1^2 + \frac{1}{2}\acute{a}_2^2 + \mu\,(\acute{a}_1 a_2 - \acute{a}_2 a_1) + \frac{1}{2}\mu^2\,(a_1^2 + a_2^2)\,. \quad (2.10)$$

We can also solve for the longitudinal coordinate $\acute{v}$ in terms of the dynamical transverse physical fields:

$$\begin{aligned} \acute{v} &= -(\acute{a}_1\Pi_1 + \acute{a}_2\Pi_2) \\ &= -\dot{a}_1 a_1' - \dot{a}_2 a_2' - \mu\,(a_2 a_1' - a_1 a_2')\,. \end{aligned} \quad (2.11)$$

The form of this last expression cannot be naively guessed and differs from the flat-space result by the H-dependent terms. This constraint, integrated over σ, produces the left-right matching condition on the physical Hilbert space.

The solutions to the equation of motion are given by

$$a = \tfrac{(a^1 + i\,a^2)}{2} = i\,e^{i\mu\sigma^+}\left(\sum_{n\in\mathbb{Z}} \frac{\tilde{a}_n}{n+\mu}\,e^{-i(n+\mu)\sigma^+} + \frac{a_n}{n-\mu}\,e^{-i(n-\mu)\sigma^-}\right) \quad (2.12)$$

$$\bar{a} = \tfrac{(a^1 - i\,a^2)}{2} = i\,e^{-i\mu\sigma^+}\left(\sum_{n\in\mathbb{Z}} \frac{\bar{\tilde{a}}_n}{n-\mu}\,e^{-i(n-\mu)\sigma^+} + \frac{\bar{a}_n}{n+\mu}\,e^{-i(n+\mu)\sigma^-}\right), \quad (2.13)$$

where we have introduced the notations, $\mu \equiv p^+ H$, and $\sigma^\pm = \tau \pm \sigma$.

The $\tilde{a}_0$ ($\bar{\tilde{a}}_0$) is the center-of-mass coordinates and should be identified with ρ ($\bar{\rho}$). Similarly we should identify a_0 ($\bar{a}_0$) with the radius, λ ($\bar{\lambda}$), of the classical trajectory. One also observes that the terms linear in τ are not allowed and thus there is no zero-mode

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momentum operators in the mode expansion above. If one takes the limit $H \to 0$, the frequency of the a_0 mode goes to zero and becomes the momentum operator of the limiting flat space.

Quantization. Upon quantizing we obtain the commutation relations for the oscillators:

$$
\begin{aligned}
[a_n, \bar{a}_m] &= \frac{1}{2}\delta_{n,-m}(n-\mu), \\
[\tilde{a}_n, \bar{\tilde{a}}_m] &= \frac{1}{2}\delta_{n,-m}(n+\mu)\,.
\end{aligned}
\tag{2.14}
$$

The reality condition gives

$$
a_n^\dagger = \bar{a}_{-n}\,; \qquad \tilde{a}_n^\dagger = \bar{\tilde{a}}_{-n}\,. \tag{2.15}
$$

When the value of μ is restricted to be $0 \le \mu < 1$ creation operators are those with negative indices, $m < 0$. Unlike the case in flat space case, the right and left moving zero-modes here are not degenerate, i.e. they have frequencies of $+\mu$ and $-\mu$ respectively. This is reflected in the mode expansion already — the right and left moving zero-modes are independent of each other.

When $\mu = N + \epsilon$ where N is a positive integer the vacuum is annihilated by a_{N+m}, $\bar{\tilde{a}}_{N+m}$, $m > 0$ and $\tilde{a}_{-N+m}, \bar{a}_{-N+m}$, $m \ge 0$. The "zero modes" are given by $a_N, \tilde{a}_{-N}$ and the corresponding classical classical solution is

$$
\frac{ie^{iN\sigma^+}}{\epsilon}\left(\tilde{a}_{-N} - a_N e^{2i\epsilon\tau}\right). \tag{2.16}
$$

When $N = 0$ this is the geodesic motion, centering at $\tilde{a}_0/\mu$ and with a radius a_0/μ, oscillating in time at frequency μ. But for $N \ne 0$ this describes the motion of a "long string",[4]

$$
e^{iN\sigma}\left(\frac{\tilde{a}_{-N}}{\epsilon}e^{iN\tau} - \frac{a_N}{\epsilon}e^{iN\tau}e^{2i\epsilon\tau}\right) \tag{2.17}
$$

i.e. a 2-dimensional surface winding N times around the origin. It envelopes the geodesic — centered at $\tilde{a}_{-N}/\epsilon$ and with a radius of a_N/ϵ — and oscillates with a slow frequency ϵ. What we are witnessing here is a dynamical dielectric effect [21] such that the light-cone momentum is transmuted into winding number under the influence of the H field: For every increase of the light-cone momentum by unit value of $\frac{1}{2\pi\alpha l^2}H^{-1}$ the winding number of the ground state also increases by one. The $\mu = 0$ case, which correctly reproduce the flat space linear motion, has been discussed above in the previous subsection.

Finally the quantum hamiltonian can be obtained by substituting the string mode expansions into the classical expression and normal-ordering:

$$
\mathcal{H}_{lc} = \sum_{n\in\mathbb{Z}}\left(:a_n\bar{a}_{-n}:\,\frac{n}{n-\mu} + :\tilde{a}_n\bar{\tilde{a}}_{-n}:\,\frac{n+2\mu}{n+\mu}\right). \tag{2.18}
$$

[4]See [8, 22] for a general definition and [11, 6] for an application of this notion to the Nappi-Witten model.

2.3 Wavefunctions

In this section we follow on to the semi-classical analysis with the construction of the wave functions on the group manifold. The classical picture will again provide the guiding principle: we will choose the wavefunctions to approximate the classical trajectories as close as possible. As a result the Landau-like orbits will translate into the coherent wavefunctions in quantum mechanics. This exercise provides us with more intuition in the propagation and interaction of particles on the Nappi-Witten space. It will also be our starting point for writing down the vertex operator in section 4.2. Appendix B contains the group theoretical analysis and the representation theory of the Nappi-Witten algebra. Readers who are not familiar with these aspects of the model should consult it at this point.

The quantum wavefunctions are eigenfunctions of the covariant laplacian operator on the group manifold:

$$\Delta = \frac{\partial^2}{\partial a_1^2} + \frac{\partial^2}{\partial a_2^2} + 2\,\frac{\partial}{\partial u}\,\frac{\partial}{\partial v} + \frac{1}{4}(a_1^2 + a_2^2)\,\frac{\partial^2}{\partial v^2} + \left(a_1\,\frac{\partial}{\partial a_2} - a_2\,\frac{\partial}{\partial a_1}\right)\frac{\partial}{\partial v}\,. \tag{2.19}$$

The laplacian is equal to the Casimir for the left or right symmetry generators (eq. (2.5))

$$\begin{aligned}\Delta &= 2\,J_L T_L + \frac{(J_L^+ J_L^- + J_L^- J_L^+)}{2}\\ &= 2\,J_R T_R + \frac{(J_R^+ J_R^- + J_R^- J_R^+)}{2}\,.\end{aligned} \tag{2.20}$$

This implies that the eigenfunctions of the laplacian with eigenvalues

$$\mathcal{C} = -2\,p^+\left(p^- + \frac{1}{2}\right) \tag{2.21}$$

are given by the matrix elements of g in the representations V^{p^+,p^-} (or in the conjugate representation, $\widetilde{V}^{p^+,p^-}$). In our convention (see appendix B.1) V^{p^+,p^-} representation has the highest weight with respect to J, whereas $\widetilde{V}^{p^+,p^-}$ has the lowest weight. They share the same Casimir given by $\mathcal{C}$. Let us denote matrix element of g in the coherent state basis of V^{p^+,p^-} by[5]

$$\phi^{p^+,p^-}_{\bar\rho,\lambda}(g) = \langle\rho|g|\lambda\rangle\,. \tag{2.22}$$

Using the definition of the group element eq. (2.2) and that of the coherent states we get

$$\phi^{p^+,p^-}_{\bar\rho,\lambda}(g) = e^{ip^+v+ip^-u}e^{-p^+a\bar a}\exp\left[2p^+(a\lambda e^{-iu} - \bar\rho\bar a) + 2p^+\bar\rho\lambda e^{-iu}\right]. \tag{2.23}$$

The ground state wave functional is a plane wave state positioned at $a = 0$ in the transverse plane

$$\phi^{p^+,p^-}_{0,0}(g) = e^{ip^+v+ip^-u}e^{-p^+a\bar a}\,. \tag{2.24}$$

Normalized and written in a more suggestive way

$$\phi^{p^+,p^-}_{\bar\rho,\lambda}(a,u,v) = e^{ip^+v+ip^-u}e^{-p^+|\bar a-\lambda e^{-iu}+\rho|^2}e^{p^+a(\rho+\lambda e^{-iu})-c.c.}e^{-p^+(\bar\lambda\rho e^{iu}-\text{c.c.})}\,, \tag{2.25}$$

[5]Note that the wavefunction obtained this way is not normalized

the wave functional takes the form of a gaussian centered around $\bar{a} = -\rho + \lambda e^{-iu}$. Moreover, it is a plane wave in v of momentum p^+; and the semi-classical momentum in the transverse plane, p^a, is given by $\rho + \lambda e^{iu}$. The direction u has a more complicated semi-classical momentum: $p^u = p^- - \lambda(a + \bar{\rho})e^{-iu} + c.c.$ All these results agree with the geodesics analysis in section 2.1.

Since the representation $\widetilde{V}^{p^+,p^-}$ is conjugate to V^{p^+,p^-}, the conjugate wave functional is related to the original wave functional by complex conjugation

$$\widetilde{\phi}^{p^+,p^-}_{\bar{\rho},\lambda}(g) = \overline{\phi^{p^+,p^-}_{\rho,\bar{\lambda}}(g)}\,. \tag{2.26}$$

Explicitly this reads

$$\widetilde{\phi}^{p^+,p^-}_{\bar{\rho},\lambda}(g) = e^{-ip^+v - ip^-u} e^{-p^+ a\bar{a}} e^{-p^+\lambda\bar{\lambda}} e^{-p^+\rho\bar{\rho}} \exp\left[2\,p^+(\bar{a}\lambda e^{+iu} - \bar{\rho}a + \bar{\rho}\lambda e^{+iu})\right]. \tag{2.27}$$

This corresponds to a wave moving backward in the light-cone time and centred around $a = \lambda e^{iu} - \rho$. Finally we can construct the wave functions corresponding to the $p^+ = 0$ representation (See appendix B).

The wave functions we have constructed are eigenvectors under the action of the lowering isometry generators $J^{\pm}_{L|R}$ (eq.2.5):

$$\begin{aligned} J^-_L\,\phi^{p^+,p^-}_{\bar{\rho},\lambda}(g) &= (2\,p^+\bar{\rho})\phi^{p^+,p^-}_{\bar{\rho},\lambda}(g)\,, & J^+_R\,\phi^{p^+,p^-}_{\bar{\rho},\lambda}(g) &= (2\,p^+\lambda)\phi^{p^+,p^-}_{\bar{\rho},\lambda}(g)\,, \\ J^+_L\,\widetilde{\phi}^{p^+,p^-}_{\bar{\rho},\lambda}(g) &= (2\,p^+\bar{\rho})\widetilde{\phi}^{p^+,p^-}_{\bar{\rho},\lambda}(g)\,, & J^-_R\,\widetilde{\phi}^{p^+,p^-}_{\bar{\rho},\lambda}(g) &= (2\,p^+\lambda)\widetilde{\phi}^{p^+,p^-}_{\bar{\rho},\lambda}(g)\,. \end{aligned} \tag{2.28}$$

Together $\phi^{p^+,p^-}_{\bar{\rho},\lambda}$, $\widetilde{\phi}^{p^+,p^-}_{\bar{\rho},\lambda}$ form a complete basis of normalisable (L^2 integrable) solutions of the wave equations[6]

$$\Delta\phi = -2\,p^+(p^- + 1/2)\phi\,, \qquad \partial_v\phi = \pm ip^+\phi \qquad \text{where} \quad p^+ > 0\,. \tag{2.29}$$

It will be very important for us to note that $\phi^{-p^+,-(p^-+1)}_{\bar{\rho},\lambda}(g)$ is also solution of (eq. (2.29)). This solution is well defined as a function on the group, however it contains a factor $\exp(2p^+a\bar{a})$ which is unbounded. It is hence not normalisable and cannot be taken as a wave functional corresponding to the representation $\widetilde{V}^{p^+,p^-}$. However one can perform integral transform of this solution

$$\int d^2\lambda\; e^{2p^+\lambda\rho}\, e^{2p^+\bar{\lambda}\bar{\rho}}\, \phi^{-p^+,-p^--1}_{\bar{\lambda}\lambda}\,, \tag{2.30}$$

which is proportional to the conjugate wave function (eq.(2.27)).

Clebsh-Gordan coefficients. A lot of important information, mathematical as well as physical, is encoded in the way the product of two wavefunctions decomposes as a linear sum of wavefunctions. On the mathematical side it contains all the information we need about the tensor product of representations and the recoupling coefficient involved. On the physical side we can read out what the conservation rules are and how two waves interact in our curved background. We are interested in the multiplicative properties of

[6] See appendix B.

the normalized wave functions

$$\phi^{p^+,p^-}_{\bar\rho,\lambda}(g) = \sqrt{\frac{p^+}{\pi}}\, e^{-p^+\lambda\bar\lambda} e^{-p^+\rho\bar\rho} e^{ip^+(v+ia\bar a)+ip^-u}\, e^{2p^+a\lambda e^{-iu}}\, e^{-2p^+\bar\rho\bar a}\, e^{2p^+\bar\rho\lambda e^{-iu}}\,.$$

It is easy to see that the product is given by

$$\begin{aligned}\phi^{p_1^+,p_1^-}_{\bar\rho_1,\lambda_1}(g)\,\phi^{p_2^+,p_2^-}_{\bar\rho_2,\lambda_2}(g) &= \phi^{p_3^+,p_3^-}_{\bar\rho_3,\lambda_3}(g)\sqrt{\frac{p_1^+p_2^+}{\pi\, p_3^+}}\,\exp\left[e^{-iu}(2\,p_1^+\bar\rho_1\lambda_1 + 2\,p_2^+\bar\rho_2\lambda_2 - 2\,p_3^+\bar\rho_3\lambda_3)\right]\times \\ &\times \exp\left[-\frac{p_1^+p_2^+}{p_3^+}(\lambda_1-\lambda_2)(\bar\lambda_1-\bar\lambda_2)\right]\exp[-\frac{p_1^+p_2^+}{p_3^+}(\rho_1-\rho_2)(\bar\rho_1-\bar\rho_2)\Big]\,,\end{aligned} \tag{2.31}$$

where we denote $p_3^\pm = p_1^\pm + p_2^\pm$, $p_3^+\lambda_3 = p_1^+\lambda_1 + p_2^+\lambda_2$, $p_3^+\rho_3 = p_1^+\rho_1 + p_2^+\rho_2$ as the momentum, position and radius of the resulting wavefunction. The formulae for the addition of "position" and "radius" follow also from the fact that $p^+\lambda$ and $p^+\rho$ are charges associated with the left- and right-acting isometries respectively. Intuitively it is easier to explain as follows. In light-cone frame p^+ is the "effective mass" for motion in the transverse directions. Thus the above formula simply states that when two particles coalesce into one, the resulting particle appears at the centre of mass position. The same intuitive explanation can be offered to the "radius" addition rule if we go to a rotating frame where the roles of radius and position are interchanged.

The term in the exponent can be evaluated to be

$$\frac{p_1^+p_2^+}{p_3^+}(\lambda_1-\lambda_2)(\bar\rho_1-\bar\rho_2)\,e^{-iu}\,. \tag{2.32}$$

If we Taylor expand the exponential and utilize the fact that $e^{-iu}\phi^{p^+,p^-}_{\bar\rho,\lambda} = \phi^{p^+,p^--1}_{\bar\rho,\lambda}$, we get a simple result

$$\begin{aligned}\phi^{p_1^+,p_1^-}_{\bar\rho_1,\lambda_1}(g)\,\phi^{p_2^+,p_2^-}_{\bar\rho_2,\lambda_2}(g) &= \frac{1}{\sqrt{\pi}}\exp\left(-\frac{p_1^+p_2^+}{p_3^+}|\rho_1-\rho_2|^2\right)\exp\left(-\frac{p_1^+p_2^+}{p_3^+}|\lambda_1-\lambda_2|^2\right)\times \\ &\times\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{p_1^+p_2^+}{p_3^+}\right)^{n+1/2}(\lambda_1-\lambda_2)^n(\bar\rho_1-\bar\rho_2)^n\,\phi^{p_3^+,p_3^--n}_{\bar\rho_3,\lambda_3}(g)\end{aligned} \tag{2.33}$$

The wavefunction overlap is suppressed exponentially by the separation of centres of two gaussians as one would expect. We can also read out the tensorisation rules

$$V^{p_1^+,p_1^-}\otimes V^{p_2^+,p_2^-} = \sum_{n=0}^{+\infty} V^{p_1^++p_2^+,p_1^-+p_2^--n}\,, \tag{2.34}$$

and the Clebsh-Gordan coefficients

$$\begin{aligned}C^{p_1^\pm p_2^\pm\lambda_3}_{\lambda_1\lambda_2p_3^\pm} &= \sum_{n>0}\delta(p_1^++p_2^+-p_3^+)\;\delta(p_1^-+p_2^--p_3^--n)\delta^2(p_1^+\lambda_1+p_2^+\lambda_2-p_3^+\lambda_3)\times \\ &\times\frac{1}{\sqrt{n!}}\exp\left[-\frac{p_1^+p_2^+}{p_3^+}|\lambda_1-\lambda_2|^2\right]\left(\frac{p_1^+p_2^+}{p_3^+}\right)^{\frac{n}{2}+\frac{1}{4}}(\lambda_1-\lambda_2)^n\,.\end{aligned} \tag{2.35}$$

Here we see that the p^- momentum is not conserved. The change is in integral steps proportional to H, and that the probability distribution is Poisson. Thus for large values of separation/radii of the incoming particles compared to string scale, we may wish to approximate the Poisson distribution with the normal distribution. The average shift in the value of p^- is

$$\langle \delta p^- \rangle = \frac{p_1^+ p_2^+}{p_3^+} (\lambda_1 - \lambda_2)(\bar{\rho}_1 - \bar{\rho}_2)$$

and the width of the distribution is equal to the square root of $\langle \delta p^- \rangle$.

3. Nappi-Witten current algebra and its free field realization

Here we start solving our model in a covariant way. To do that we propose a new free field realization using the $\beta - \gamma$ system in section 3.1. Next we construct the currents and compute the corresponding algebra. In section 3.2 we identify the original group coordinates in terms of the Wakimoto free fields and hence provide a geometric interpretation for latter. Finally to pave the way for constructing the vertex operators we prove that the field identifications, done at the classical level, can be promoted to operator relations: The OPEs of spacetime fields with the symmetry currents are given by the action of the isometries.

3.1 Nappi-Witten WZW model

We now present a new realization of the current algebra in terms of free fields, comprised of a pair of null bosonic fields u, $\tilde{v}$ and the bosonic ghost system β, γ. A generic group element is written in terms of these fields

$$g = e^{\gamma J^-} e^{uJ + \tilde{v}T} e^{\bar{\gamma} J^+} . \tag{3.1}$$

The inverse is very simple in this representation, and simplifies many computations:

$$g^{-1} = e^{-\bar{\gamma}\, J^+} e^{-u\, J - \tilde{v}\, T} e^{-\gamma\, J^-} . \tag{3.2}$$

We can identify these variables with the geometric coordinates on the group

$$\gamma = \bar{a} \ , \qquad \bar{\gamma} = e^{-iu}\, a \, , \qquad \tilde{v} = v + i\, a\, \bar{a} \, . \tag{3.3}$$

It is straightforward to obtain the corresponding WZW action in terms of these variables:

$$S = \frac{1}{2\pi} \int d^2 z \, (\, \partial u \, \bar{\partial} \tilde{v} + 2\, e^{iu}\, \bar{\partial}\gamma \, \partial \bar{\gamma}) \, . \tag{3.4}$$

The metric and B-field are:

$$ds^2 \ = 4\, e^{iu} d\gamma \, d\bar{\gamma} + 2\, du \, d\tilde{v} \tag{3.5}$$

$$B \ = 2 e^{iu} d\gamma \wedge d\bar{\gamma} + du \wedge d\tilde{v} \, . \tag{3.6}$$

We introduce two subsidiary fields β and $\bar{\beta}$, which are the canonical momenta associated with γ and $\bar{\gamma}$. In the usual first order formalism the action takes the form:

$$S = \frac{1}{2\pi}\int\left(\partial u\,\bar{\partial}\tilde{v} + \bar{\beta}\,\partial\bar{\gamma} + \beta\,\bar{\partial}\gamma - \frac{1}{2}\,e^{-iu}\,\beta\,\bar{\beta}\right). \qquad (3.7)$$

The equations of motion enforce that $\beta = 2\,e^{iu}\,\partial\bar{\gamma}$ and $\bar{\beta} = 2\,e^{iu}\,\bar{\partial}\gamma$.

This form of the action is ultimately the basis of our solution to the Nappi-Witten model. The first three terms make up the free action of the β,γ ghost system ($c = 2$) plus a pair of null light-cone fields ($c = 2$). The last term, usually called the screening charge in the literature, can be viewed as an interaction term which however cannot contribute to any loop diagrams on the worldsheet.

The symmetry currents $J_L(z) = g\,\partial g^{-1}$ and $J_R(\bar{z}) = g^{-1}\,\bar{\partial}g$ can also be conveniently rewritten in terms of the first-order fields:

$$\begin{aligned} J_L^-(z) &= -\beta & J_R^-(\bar{z}) &= 2\,(\bar{\partial}\bar{\gamma} + i\,\bar{\partial}u\,\bar{\gamma}) \\ J_L^+(z) &= -2\,(\partial\gamma + i\,\partial u\,\gamma) & J_R^+(\bar{z}) &= \bar{\beta} \\ J_L(z) &= -(\partial\tilde{v} - i\,\beta\,\gamma) & J_R(\bar{z}) &= \bar{\partial}\tilde{v} - i\,\bar{\beta}\,\bar{\gamma} \\ T_L(z) &= -\partial u & T_R(\bar{z}) &= \bar{\partial}u\,. \end{aligned} \qquad (3.8)$$

From these expressions one may suspect that β and γ are purely left-moving, whereas $\bar{\beta}$ and $\bar{\gamma}$ are purely right-moving. This is indeed the case and will be clear from the field decompositions in (3.12).

Contracting free fields according to

$$\begin{aligned} u(z)\,\tilde{v}(w) &\sim \ln(z-w) \\ \beta(z)\,\gamma(w) &\sim \frac{1}{z-w} \end{aligned} \qquad (3.9)$$

the currents satisfy the correct OPEs,

$$\begin{aligned} J(z)\,J^{\pm}(w) &\sim \pm i\,\frac{J^{\pm}(z)}{z-w} \\ J^{+}(z)\,J^{-}(w) &\sim \frac{2}{(z-w)^2} + 2\,i\,\frac{T(z)}{z-w} \\ T(z)\,J(w) &\sim \frac{1}{(z-w)^2}\,. \end{aligned} \qquad (3.10)$$

The Sugawara energy-momentum tensor also admits a simple form:

$$\mathcal{T}(z) = J(z)\,T(z) + \frac{J^{+}(z)\,J^{-}(z)}{2} = \beta(z)\,\partial\gamma(z) + \partial u\,\partial\tilde{v}\,. \qquad (3.11)$$

Readers who are interested in the amplitudes may wish to skip ahead to section 5.

3.2 Free field representation of the coordinates

We have seen in the previous section how the Wakimoto free field representation comes naturally from the Nappi-Witten action when it is written in terms of the new coordinates and in first order formalism. However we have to pay a price for the algebraic simplicity of this representation, we have lost the geometrical interpretation because the ghost β is not a coordinate field in spacetime. In this section we will recover the geometrical interpretation of the free fields by expressing them in terms of the spacetime fields. Such geometrical representation for the $\gamma - \beta$ system has been proposed in the SU(2) case [23, 24].

The Wakimoto free fields consist of the set of holomorphic free fields $u_L(z), \tilde{v}_L(z), \gamma_L(z), \beta_L(z)$ and antiholomorphic free fields $u_R(\bar{z}), \tilde{v}_R(\bar{z}), \bar{\gamma}_R(\bar{z}), \bar{\beta}_R(\bar{z})$. We want to understand the relation of these fields with the spacetime fields $a(z,\bar{z}), \bar{a}(z,\bar{z}), u(z,\bar{z}), v(z,\bar{z})$. This can be done if we look at the classical solution of the Nappi-Witten model. It is well known that a general solution to a WZW model is given by a product of left and right movers $g(z,\bar{z}) = g_L(z)\, g_R(\bar{z})$. If we introduce the left- and right-moving fields $u_{L|R}, \tilde{v}_{L|R}, \gamma_{L|R}, \bar{\gamma}_{L|R}$, parametrizing group elements as in (3.1), then this solution can be explicitly written

$$\begin{aligned} a(z,\bar{z}) &= e^{+iu_L(z)}\, [\bar{\gamma}_L(z) + e^{iu_R(\bar{z})}\bar{\gamma}_R(\bar{z})]\,, \\ \bar{a}(z,\bar{z}) &= e^{-iu_L(z)}\, [e^{iu_L(z)}\gamma_L(z) + \gamma_R(\bar{z})]\,, \\ u(z,\bar{z}) &= u_L(z) + u_R(\bar{z})\,, \\ v(z,\bar{z}) + i\, a\, \bar{a}(z,\bar{z}) &= \tilde{v}_L(z) + \tilde{v}_R(\bar{z}) + 2\, i\, \bar{\gamma}_L(z)\, \gamma_R(\bar{z})\,. \end{aligned} \tag{3.12}$$

It is then easy to check that this is a solution of the Nappi-Witten equations of motion. The fields $\bar{\gamma}_R, \gamma_L$ do not possess any monodromy when going around the point $z = 0$, but the fields $\bar{\gamma}_L, \gamma_R$ do.

Because the current algebra 3.8 has been realized by using only γ_L, β_L and $\bar{\gamma}_R, \bar{\beta}_R$, we would like to entirely purge the dynamical field content of the theory of the fields $\bar{\gamma}_L, \gamma_R$. This is possible to achieve by inverting the on shell relations satisfied by the ghost fields $\beta_L(z), \bar{\beta}_R(\bar{z})$

$$\begin{aligned} \beta_L(z) &= 2\, e^{iu_L}\, \partial_z \bar{\gamma}_L(z)\,, \\ \bar{\beta}_R(\bar{z}) &= 2\, e^{iu_R}\, \bar{\partial}_{\bar{z}} \gamma_R(\bar{z})\,. \end{aligned} \tag{3.13}$$

We propose the following contour integrals in order to express the two remaining coordinates $\bar{\gamma}_L, \gamma_R$ in terms of the Wakimoto free fields

$$\begin{aligned} \bar{\gamma}_L(z) &= \frac{1}{\sin \pi p^+} \oint_{-\pi}^{+\pi} \frac{d\sigma}{2}\, e^{-iu_L(ze^{i\sigma})}\, ze^{i\sigma}\, \beta_L(ze^{i\sigma})\,, \\ \gamma_R(\bar{z}) &= \frac{1}{\sin \pi p^+} \oint_{-\pi}^{+\pi} \frac{d\sigma}{2}\, e^{-iu_R(\bar{z}e^{i\sigma})}\, \bar{z}e^{i\sigma}\, \bar{\beta}_R(\bar{z}e^{i\sigma})\,, \end{aligned} \tag{3.14}$$

where p^+ is the monodromy of the chiral fields $u_L(z), u_R(\bar{z})$ when going around 0: $ip^+ = \frac{1}{2\pi} \oint dz\, \partial u(z)$.

3.3 Quantum free field

Our goal in this section is to show that the previous representation of the space time fields can be extended to the quantum level. Namely, we want to show that the OPE of the currents with the spacetime fields can be interpreted in terms of the spacetime isometry. This means that we now have to consider the fields $u_{L|R}, \tilde{v}_{L|R}, \gamma_L, \bar{\gamma}_R, \beta_L, \bar{\beta}_R$ to be quantum operators. The integrals (3.14) above define the composite operators $\bar{\gamma}_L, \gamma_R$. No operator ordering is needed because $u(z)$ commutes with itself since it is a null direction and it also commutes with β. We can now use the expression (3.12) to define the coordinate fields $a, \bar{a}, u, v$ as quantum fields. The coordinate fields involve products of free fields at the same point, so we have to be a little bit cautious in their definition. As before u is null and commutes with $\gamma_{L|R}, \bar{\gamma}_{L|R}$, so the only problematic product is the one of a with $\bar{a}$ since it involves a product of β with γ. We take care of this issue in the usual way by normal ordering the β and γ. This means that the spacetime fields defined by (3.12), (3.14) can be considered as quantum operators. From the definition of the currents (3.8) in terms of the Wakimoto free fields we can easily compute the OPEs of the currents with the free fields.

Let $F(a, \bar{a}, u, v)(z, \bar{z})$ be a linear functional of the fields (we take it to be linear in order to avoid operator ordering problems). For the left-moving fields

$$
\begin{aligned}
T_L(z)\ F(w, \bar{w}) &\sim -\frac{1}{z-w}\ \partial_v F(w, \bar{w})\,, \\
J_L(z)\ F(w, \bar{w}) &\sim -\frac{1}{z-w}\ (\partial_u + i(a\partial_a - \bar{a}\partial_{\bar{a}}))\, F(w, \bar{w})\,, \\
J_L^+(z)\ F(w, \bar{w}) &\sim -\frac{1}{z-w}\ (\partial_a + i\bar{a}\partial_v)\, F(w, \bar{w})\,, \\
J_L^-(z)\ F(w, \bar{w}) &\sim -\frac{1}{z-w}\ (\partial_{\bar{a}} - ia\partial_v)\, F(w, \bar{w})\,.
\end{aligned}
\tag{3.15}
$$

For the right-moving fields

$$
\begin{aligned}
T_R(\bar{z})\ F(w, \bar{w}) &\sim \frac{1}{\bar{z}-\bar{w}}\ \partial_v F(w, \bar{w})\,, \\
J_R(\bar{z})\ F(w, \bar{w}) &\sim \frac{1}{\bar{z}-\bar{w}}\ \partial_u F(w, \bar{w})\,, \\
J_R^+(\bar{z})\ F(w, \bar{w}) &\sim \frac{1}{\bar{z}-\bar{w}}\ e^{iHu}(\partial_a - i\,\bar{a}\,\partial_v) F(w, \bar{w})\,, \\
J_R^-(\bar{z})\ F(w, \bar{w}) &\sim \frac{1}{\bar{z}-\bar{w}}\ e^{-iHu}(\partial_{\bar{a}} + i\,a\,\partial_v) F(w, \bar{w})\,.
\end{aligned}
\tag{3.16}
$$

On the RHS of these equations we recognize the action of the space isometries already discussed in (2.5). This means that our construction of the spacetime coordinate fields in terms of free fields is indeed the most natural one. This construction is very different from the previous construction of Kiritsis and Kounnas [9]. Moreover the geometrical interpretation of all the free fields, including the β field is now clear, which is usually one of the weak points of the Wakimoto free field representation.

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The proof of the OPEs (3.15), (3.16) is obtained by a direct and systematic computation. Let us outline it for the right algebra. Using (3.9) the nontrivial OPEs of $u, v, \bar{\gamma}$ with J are readily given by(we only list the right sector here):

$$\begin{aligned} &J_R(\bar{z})\ \bar{\gamma}_R(\bar{w}) \sim -\frac{i\,\bar{\gamma}_R(\bar{w})}{\bar{z}-\bar{w}}\,, && J_R(\bar{z})\ u_R(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}}\,, \\ &J_R^-(\bar{z})\ \tilde{v}_R(\bar{w}) \sim \frac{2\,i\,\bar{\gamma}_R(\bar{w})}{\bar{z}-\bar{w}}\,, && T_R(\bar{z})\ \tilde{v}_R(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}}\,, \\ &J_R^+(\bar{z})\ \bar{\gamma}_R(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}}\,. \end{aligned} \tag{3.17}$$

To compute the OPE of the currents with γ_R one first computes the OPE of the currents with $\bar{\partial}\gamma_R = 1/2\, e^{-iu_R}\,\bar{\beta}_R(\bar{z})$ and checks that this object has a trivial OPE with all the currents except for J_R^-

$$J_R^-(\bar{z})\ \frac{1}{2}e^{-iu_R}(\bar{w})\bar{\beta}_R(\bar{w}) \sim \bar{\partial}_{\bar{w}}\left(\frac{e^{-iu_R(\bar{w})}}{\bar{z}-\bar{w}}\right). \tag{3.18}$$

Note that in the literature on Wakimoto free fields this field is called the 'screening current' and γ_R the 'screening charge'. From the previous OPE one gets that the only nontrivial OPE of J with γ_R is given by

$$J_R^-(\bar{z})\ \gamma_R(\bar{w}) \sim \frac{e^{-iu_R(\bar{w})}}{\bar{z}-\bar{w}}\,. \tag{3.19}$$

4. Vertex operators and their construction

We have constructed a Wakimoto representation of the current algebra in terms of free fields. We now want to use this free field representation to define the vertex operators and compute the correlation functions of the theory. This computation will give us a very simple and general formula that we will consider more to be a conjecture than a full proof. we will in the next section make use of the free field representation to construct the vertex operators as operators in the Hilbert space and compute their matrix elements. Before going on with the computation let us give a description of the Nappi-Witten Hilbert space and recall some general and well known facts about conformal field theory and vertex operators.

4.1 Nappi-Witten Hilbert space and vertex operators

One of the main properties of a two-dimensional conformal field theory is the fact that there is a general correspondence between states and operators. This correspondence leads to the key notion of vertex operators as follows. Given a state $|\phi\rangle$ in the Hilbert space, there is a unique operator $V_{|\phi\rangle}(z,\bar{z})$ such that

$$V_{|\phi\rangle}(0)\ |0\rangle = |\phi\rangle\,, \tag{4.1}$$

with $|0\rangle$ the $sl(2,C)$ invariant vacuum. In order to describe the vertex operators we first have to describe the Hilbert space of our theory. We have seen that the Quantum mechanical Hilbert space is given by the space of wave functions

$$L^2(G) = \int_0^{+\infty} dp^+ \int_{-\infty}^{+\infty} dp^- \left(V^{p^+,p^-} \otimes \widetilde{V}^{p^+,p^-} \oplus \widetilde{V}^{p^+,p^-} \otimes V^{p^+,p^-} \right). \tag{4.2}$$

From the point of view of the CFT the Hilbert space is constructed out of irreducible representations of the affine algebra. It is well known that given a representation V^{p^+,p^-} of the Lie algebra we can construct an highest weight representation of the current algebra denoted $\mathcal{F}^{p^+,p^-}$. These representations are generated by action of the negative modes currents J^a_{-n} on the vectors $|\lambda\rangle \in V^{p^+,p^-}$ which are all annihilated by positive modes currents $J^a_n\,|\lambda\rangle = 0,\ n > 0$. Due to the timelike signature of spacetime these representations are not unitary. However in the Nappi-Witten model one can check that, after imposition of the string mass shell conditions $(L_0 - 1)\,|\psi\rangle = 0; L_n\,|\psi\rangle = 0, n > 0$, these representations are ghost free as long as $p^+ < 1$.[7] This is already clear in the light-cone analysis of the model (section 2.2). We will see that it is not possible to restrict to the sector with $p^+ < 1$ since long-string states appear in the factorization of four-point amplitude (section 5.2). Similar analysis had been done for strings in AdS_3 [26]. This shows that long-strings are part of the physical spectrum. If $p^+ = 1$ the representation contains null states and if $p^+ > 1$ the highest weight representation contains additional negative-norm states.[8] In order to construct ghost free representation of the current algebra with $p^+ > 1$ we have to consider spectral flowed representation [22]. First let us remark that the Nappi-Witten current algebra possesses an automorphism [11, 6] which is

$$S_w(J^\pm_n) = J^\pm_{n\pm w}\,, \qquad S_w(T_n) = T_n + i\,w\,\delta_{n,0}\,, \qquad S_w(J_n) = J_n\,. \tag{4.3}$$

Spectral flowed representations $\mathcal{F}^{p^+,p^-}_w$ are defined as the highest weight representations of the algebra obtained after action of this automorphism.[9] The necessity of spectral flowed representation is clear from the semi-classical analysis where they correspond to string winding around the transverse plane[10][22, 11]. We have also seen in section 2.2 that they naturally appear in the spectrum of the light-cone hamiltonian [17]. Overall, this leads to a Hilbert space

$$\mathcal{H} = \sum_{w=0}^{\infty} \int_0^1 dp^+ \int_0^{+\infty} dp^- \left(\mathcal{F}^{p^+,p^-}_w \otimes \widetilde{\mathcal{F}}^{p^+,p^-}_w \oplus \widetilde{\mathcal{F}}^{p^+,p^-}_w \otimes \mathcal{F}^{p^+,p^-}_w \right). \tag{4.4}$$

Given a state $|\bar{\lambda},\lambda\rangle \in \widetilde{V}^{p^+,p^-} \otimes V^{p^+,p^-}$ we can associate to it an highest weight state of

[7] See [25] for a clear exposition of similar facts in AdS_3.

[8] One can directly check that the square norm of $J^-_{-1}\,|0\rangle$ is given by $2\,(1-p^+)$.

[9] A formal proof that these representations are ghost free is expected to be very similar to the AdS$_3$ case but is not yet available in the literature, see however [27].

[10] The automorphism comes from the following action in the loop group $g(z,\bar{z}) \to z^{iwJ}\,g\,\bar{z}^{-iwJ}$.

$\widetilde{\mathcal{F}}_w^{p^+,p^-} \otimes \mathcal{F}_w^{p^+,p^-}$ if we identify w with the integer part of p^+. A general state in the Hilbert space is obtained by repeated action of the negative modes of the currents J^i_{-n} on a highest weight state.

The vectors labelled by nonzero integers ω are long-string states. In these sectors the spectra are labelled by $0 \leq p^+ < 1$, $p^- \in \mathbb{R}$. The state corresponding to $p^+ = 0$ has a special status: it corresponds to a long-string vaccuum. This long-string state, as well as those related to $p^+ = 0$ state by spectrum flow [6], experiences no potential in the transverse directions. Therefore the representation they lie in are different from the massive long string states with $p^+ \neq 0$. If we consider, as we do in this paper, wave packets (labelled by p^+) localized in the transverse plane, these states appearing as one point in a continous spectrum do not play a major role. The representation for these states are drastically different from the massive p^+ representations, hence the construction of vertex operators for these states is modified, and will the subject of a future publication. At present we concentrate only on massive $p^+ \neq 0$ states.

4.2 Vertex operators

In this section we want to construct the primary vertex operators which are the vertex operators associated with the highest weight states. From the previous section discussion we now understand that there is a one to one correspondence between the wave functions $\phi^{p^+,p^-}_{\bar{\lambda},\lambda}(g) \in \widetilde{V}^{p^+,p^-} \otimes V^{p^+,p^-}$, and the primary vertex operators $V^{p^+,p^-}_{\bar{\lambda},\lambda}(z,\bar{z})$.[11] We now want to construct this vertex operator restricting ourselves to the case $0 < p^+ < 1$ of unflowed states. We have seen in section 3.3 that the representation of the coordinates in terms of free fields introduced in section 3.2 satisfy the expected OPE with the currents. It is interesting to reexpress the wave functional $\phi^{p^+,p^-}_{\bar{\lambda},\lambda}(g)$ in terms of the free fields, after substitution we obtain

$$V^L_{\bar{\lambda}}(z)\; V^R_{\lambda}(\bar{z})\; e^{-2p^+ S_{\bar{\lambda}\lambda}(z,\bar{z})}\,, \tag{4.5}$$

where

$$\begin{aligned} V^L_{\bar{\lambda}}(z) &= e^{ip^+\tilde{v}_L}\, e^{ip^- u_L}\, e^{-2p^+\bar{\lambda}\gamma_L}\,,\\ V^R_{\lambda}(\bar{z}) &= e^{ip^+\tilde{v}_R}\, e^{ip^- u_R}\, e^{2p^+\lambda\bar{\gamma}_R}\,, \end{aligned} \tag{4.6}$$

and

$$S_{\bar{\lambda}\lambda}(z,\bar{z}) = (\gamma_R - \lambda e^{-iu_R})(\bar{\gamma}_L + \bar{\lambda} e^{-iu_L})\,. \tag{4.7}$$

We want to promote this wave functional to a quantum vertex operator, this is done by normal ordering the operators. We denote as usual $:O:$ the normal ordering of an operator and define

$$V^{p^+,p^-}_{\bar{\lambda},\lambda}(z,\bar{z}) = :V^L_{\bar{\lambda}}(z)\colon :V^R_{\lambda}(\bar{z})\colon :e^{-2p^+ S_{\bar{\lambda}\lambda}(z,\bar{z})(z,\bar{z})}:. \tag{4.8}$$

[11]Similarly one can associate a conjugate vertex operator $\widetilde{V}^{p^+,p^-}_{\bar{\lambda},\lambda}(z,\bar{z})$ to the conjugate wave functional $\widetilde{\phi}^{p^+,p^-}_{\bar{\lambda},\lambda}(g) \in V^{p^+,p^-} \otimes \widetilde{V}^{p^+,p^-}$.

One can now check that *both* $V^{p^+,p^-}_{\bar\lambda,\lambda}(z,\bar z)$ and $:V^L_{\bar\lambda}(z):$ satisfy the following OPE with the currents:

$$
\begin{aligned}
T_L(z)\; V_{\bar\lambda}(w) &\sim \frac{-i\,p^+}{z-w}\, V_{\bar\lambda}(w)\,,\\
J_L(z)\; V_{\bar\lambda}(w) &\sim \frac{i\,(-p^- + \bar\lambda\partial_{\bar\lambda})}{z-w}\, V_{\bar\lambda}(w)\,,\\
J^+_L(z)\; V_{\bar\lambda}(w) &\sim \frac{-\partial_{\bar\lambda}}{z-w}\, V_{\bar\lambda}(w)\,,\\
J^-_L(z)\; V_{\bar\lambda}(w) &\sim \frac{2\,p^+\bar\lambda}{z-w}\, V_{\bar\lambda}(w)\,.
\end{aligned}
\tag{4.9}
$$

This OPE tells us that $V^L_{\bar\lambda}(z)$ satisfies the same OPE as the chiral vertex operator associated with the state

$$|\bar\lambda\rangle = \exp(-\bar\lambda J^+)\;|0\rangle \in \widetilde{V}^{p^+,p^-}\,. \tag{4.10}$$

Similarly *both* $V^{p^+,p^-}_{\bar\lambda,\lambda}(z,\bar z)$ and $:V^R_\lambda(\bar z):$ satisfy the same OPE with the right currents. Therefore we should associate the state

$$|\lambda\rangle = \exp(-\lambda J^-)\;|0\rangle \in V^{p^+,p^-} \tag{4.11}$$

with the chiral operator, $V^R_\lambda(\bar z)$.

Now both $V^{p^+,p^-}_{\bar\lambda,\lambda}(z,\bar z)$ and $:V^L_{\bar\lambda}(z)::V^R_\lambda(\bar z):$ satisfy the same OPE's with the currents we thus expect them to differ only by some screening operator which commutes with the currents. We have already seen that the current algebra admits a *unique* screening operator:

$$S = \frac{1}{2}\int d^2\omega\;\beta_L(\omega)\,\bar\beta_R(\bar\omega)\,e^{iu(\omega)}\,. \tag{4.12}$$

We have also seen that the Nappi-Witten action written in first order formalism eq. (3.7) is a free field theory plus an interaction term which is exactly S. This suggests that the vertex operator can be written simply as

$$V^{p^+,p^-}_{\bar\lambda,\lambda}(z,\bar z) = :V^L_{\bar\lambda}(z)\,V^R_\lambda(\bar z):\,, \tag{4.13}$$

but at the expense of the necessity to insert the screening operator into the expression for the correlation function.

5. Correlation functions

The conjecture above states that in order to compute the N-point correlation function in the interacting theory we just have to compute the free correlation function of the insertion of N free vertex operators in the presence of the screening charge. Similar approach for the case of SL(2) has been used in [35]–[41].

Thus the complete correlation function is given by

$$\Gamma_{\vec{\bar\lambda},\vec\lambda,s}(\vec z,\vec{\bar z}) = \sum_s \frac{1}{s!}\left\langle \prod_{i=1}^n V_{\lambda_i}(z_i)\;V_{\bar\lambda_i}(\bar z_i)\left(\int d^2w\;\beta(w)\,\bar\beta(\bar w)\,e^{-iu(w,\bar w)}\right)^s\right\rangle, \tag{5.1}$$

where s is a positive integer. We have to integrate over $u_0, v_0, \gamma_0, \bar{\gamma}_0$ the zero modes of the fields $u_L + u_R, v_L + v_R, \gamma_L, \bar{\gamma}_R$.[12] The full partition function can then be split after integration over the zero modes into a chiral and anti-chiral parts. The invariant measure on the group is given by

$$du_0\, dv_0\, d\gamma_0\, d\bar{\gamma}_0\, e^{iu_0}\,. \tag{5.2}$$

The integration over the u and v zero modes leads to conservation of momenta by producing delta functions

$$2\pi\delta\left(\sum_{i=1}^{n} p_i^+\right) 2\pi\delta\left(\sum_{i=1}^{n} p_i^- + 1 - s\right). \tag{5.3}$$

The integration over $\gamma_0, \bar{\gamma}_0$ also leads to momentum conservation

$$\pi^2\delta^{(2)}\left(\sum_{i=1}^{n} p_i^+ \lambda_i\right). \tag{5.4}$$

The remaining term multiplying these delta functions is

$$\int \prod_{k=1}^{s} d^2 w_k\, |\Gamma_{\vec{\lambda}}(\vec{w}, \vec{z})|^2\,, \tag{5.5}$$

where $\Gamma_{\vec{\lambda}}(\vec{w}, \vec{z})$ is a chiral conformal block given by the product of a free field conformal block and a ghost conformal block. The free field block is given by

$$\Gamma^{uv}_{\vec{\lambda}}(\vec{w}, \vec{z}) = \left\langle \prod_{i=1}^{n} e^{ip_i^+ \tilde{v}(z_i) + ip_i^- u(z_i)} \prod_{k=1}^{s} e^{-iu(w_k)} \right\rangle\,, \tag{5.6}$$

and can be easily computed by contracting the exponentials in all possible ways. There are no w_k, w_l contractions since u is a null direction. For each pair $z_i, z_j; i < j$ we get after contraction a factor $(z_i - z_j)^{-p_i^+ p_j^- - p_j^+ p_i^-}$ since the free field OPE is given by $u(z)v(w) \sim \ln(z - w)$, each pair z_i, w_k gives a factor $(w_k - z_i)^{p_i^+}$, overall we get

$$\Gamma^{uv}_{\vec{\lambda}}(\vec{w}, \vec{z}) = \prod_{i \neq j} (z_i - z_j)^{-p_i^+ p_j^-} \prod_{i=1}^{n} \prod_{k=1}^{s} (w_k - z_i)^{p_i^+}\,. \tag{5.7}$$

The ghost conformal block is given by

$$\Gamma^{\beta\gamma}_{\vec{\lambda}}(\vec{w}, \vec{z}) = \left\langle \prod_{i=1}^{n} e^{p_i^+ \lambda_i \gamma(z_i)} \prod_{k=1}^{s} \beta(w_k) \right\rangle\,. \tag{5.8}$$

It is computed by contracting each $\beta(w)$ once with each factor $\exp \lambda_i \gamma(z_i)$. Each contraction gives a contribution $\lambda(w - z)^{-1}$,

$$\Gamma^{\beta\gamma}_{\vec{\lambda}}(\vec{w}, \vec{z}) = \prod_{k=1}^{s} \left(\sum_{i=1}^{n} \frac{\lambda_i}{w_k - z_i} \right). \tag{5.9}$$

This block is homogeneous in λ_i of degree s.

[12] There is no integration over the zero modes of β_L, $\bar{\beta}_R$ since they are identified with derivatives (momenta) of coordinate fields.

We can now put everything together and since there is no coupling between different w_i we get the simple expression

$$\Gamma_{\vec{\lambda},\vec{\bar{\lambda}},s}(\vec{z},\vec{\bar{z}}) = 4\,\pi^4\;\delta\left(\sum_i p_i^+\right)\delta^{(2)}\left(\sum_{i=1}^n p_i^+\lambda_i\right)\left(\prod_{i\neq j}|z_i-z_j|^{-2p_i^+p_j^-}\right)\times$$
$$\times\sum_{s\geq 0}\;\delta\left(\sum_i p_i^- + 1 - s\right)\;\frac{1}{s!}\left(\mathcal{I}_{\vec{\lambda},\vec{p}^+}(\vec{z})\right)^s\,, \tag{5.10}$$

where

$$\mathcal{I}_{\vec{\lambda},\vec{p}^+}(\vec{z}) = \int d^2w\,\left|\Delta_{\vec{\lambda},\vec{p}^+}(w,\vec{z})\right|^2 = \int d^2w\prod_{i=1}^n|w-z_i|^{2p_i^+}\left|\sum_{i=1}^n\frac{p_i^+\lambda_i}{w-z_i}\right|^2\,. \tag{5.11}$$

In this computation we have inserted n vertex operators $V_{\bar{\lambda},\lambda}^{p^+,p^-}$ associated with the wave function $\phi_{\bar{\lambda},\lambda}^{p^+,p^-}$. These vertex operators are well defined when p^+ is positive. When p^+ is negative the appropriate vertex operator is the conjugate vertex operator $\widetilde{V}_{\bar{\rho},\rho}^{p^+,p^-}$ associated with $\widetilde{\phi}_{\bar{\lambda},\lambda}^{p^+,p^-}$. Due to the p^+ conservation rule not all p^+ momenta can be positive, at least one should be negative. This means that one should at least insert one conjugate vertex operator in the correlation function. Physically, particles with negative p^+ are necessary because they correspond to outgoing states in the path integral prescription. Fortunately, the conjugate vertex operator is related to the vertex operator we considered so far by an integral transform[13]eq. (2.30)

$$\widetilde{V}_{\bar{\rho},\rho}^{p^+,p^-} = \frac{(p^+)^2}{\pi}\int d^2\lambda\;e^{2p^+\lambda\rho}\,e^{2p^+\bar{\lambda}\bar{\rho}}\;V_{\bar{\lambda}\lambda}^{-p^+,-(p^-+1)}\,. \tag{5.12}$$

Using integration by parts in the right hand side of the OPEs (eq. (4.9)) one can easily show that these operators satisfy with the currents the OPEs of a conjugate field. This integral formula gives us the general rule needed in order to insert a conjugate vertex operator in the correlator: for each conjugate vertex operator one should integrate over λ a usual vertex operator associated with momenta $-p^+, -(p^-+1)$. In the case of one conjugate field we obtain

$$\left\langle\prod_{i=1}^{n-1}V_{\bar{\lambda}_i\lambda_i}^{p_i^+,p_i^-}(z_i)\widetilde{V}_{\bar{\rho}_n\rho_n}^{p_n^+,p_n^-}(z_n)\right\rangle = \frac{1}{\pi}\delta\left(\sum_i p_i^+\right)\left|e^{\rho_n(\sum_{i=1}^{n-1}2p_i^+\lambda_i)}\right|^2\prod_{i\neq j}|z_i-z_j|^{-2p_i^+p_j^-}\,\cdot$$
$$\cdot\sum_{s\geq 0}\;\delta\left(\sum_{i=1}^{n-1}p_i^- - p_n^- - s\right)\;\frac{1}{s!}\left(\mathcal{I}_{\vec{\lambda},\vec{p}^+}(\vec{z})\right)^s\,. \tag{5.13}$$

with $p_n^+\lambda_n = \sum_{i=1}^{n-1}p_i^+\lambda_i$ in $\mathcal{I}_{\vec{\lambda},\vec{p}^+}$.

[13] $V^{-p^+,-p^--1}$ appears instead of $V^{-p^+,-p^-}$ because the former is a solution to the wave equation with the same Casimir as V^{p^+,p^-}. See appendix C for a rigorous treatment of the Integral Transform.

5.1 Three-point function

We can now specialize our general formula to the case where there is only three points z_1, z_2, z_3. In this case one can check that

$$\begin{aligned} \mathcal{I}_{\vec{\lambda},\vec{p}^+}(\vec{z}) &= |z_1 - z_2|^{-2p_3^+} |z_1 - z_3|^{-2p_2^+} |z_2 - z_3|^{-2p_1^+} \times \\ &\times |\lambda_1 - \lambda_2|^2 (p_1^+)^2 \int d^2\tilde{w} |\tilde{w}|^{2(p_1-1)} |\tilde{w} - 1|^{2p_2^+} . \end{aligned} \tag{5.14}$$

One first uses the identity $\sum_i p_i^+ \lambda_i = 0$ to eliminate λ_3, then makes the change of variable

$$w \to \tilde{w} = \frac{(z_2 - z_3)(w - z_1)}{(z_2 - z_1)(w - z_3)}, \tag{5.15}$$

and finally uses the fact that

$$w^{p_1^+} (w-1)^{p_2^+} \left(\frac{p_1^+ \lambda_2}{w} + \frac{p_2^+ \lambda_2}{w-1} \right) \tag{5.16}$$

is a total derivative which we can subtract from the integrand. The integral can be explicitly evaluated

$$\int d^2w \ |w|^{2(p_1^+ - 1)} |w - 1|^{2(p_2^+ - 1)} = \pi \frac{\Upsilon(p_1)\Upsilon(p_2)}{\Upsilon(p_1 + p_2)}, \tag{5.17}$$

where $\Upsilon(p) \equiv \frac{\Gamma(p)}{\Gamma(1-p)}$. This function satisfies $\Upsilon(p)\Upsilon(1-p) = 1$, $\Upsilon(p+1) = -p^2 \Upsilon(p)$. Therefore, using the value of the integral and the conservation of the light cone momenta we can write this integral in a symmetric form

$$\begin{aligned} \mathcal{I}_{\vec{\lambda},\vec{p}^+}(\vec{z}) &= \pi |z_1 - z_2|^{-2p_3^+} |z_1 - z_3|^{-2p_2^+} |z_2 - z_3|^{-2p_1^+} \times \\ &\times \left| \frac{\lambda_1 - \lambda_2}{p_3^+} \right|^2 \Upsilon(p_1^+ + 1) \Upsilon(p_2^+ + 1) \Upsilon(p_3^+ + 1) . \end{aligned} \tag{5.18}$$

We can now easily read out from eq. (5.13) the three-point function

$$\begin{aligned} &\left\langle V^{p_1^+, p_1^-}_{\bar{\lambda}_1 \lambda_1}(z_1) V^{p_2^+, p_2^-}_{\bar{\lambda}_2 \lambda_2}(z_2) \widetilde{V}^{p_3^+, p_3^-}_{\bar{\rho}_3 \rho_3}(z_3) \right\rangle = \\ &= \frac{1}{\pi} |z_1 - z_2|^{-2\Delta_1 - 2\Delta_2 + 2\Delta_3} |z_1 - z_3|^{-2\Delta_1 - 2\Delta_3 + 2\Delta_2} |z_2 - z_3|^{-2\Delta_2 - 2\Delta_3 + 2\Delta_1} \times \\ &\times \delta(p_1^+ + p_2^+ - p_3^+) \ |e^{2\rho_3 (p_1^+ \lambda_1 + p_2^+ \lambda_2)}|^2 \times \\ &\times \sum_{s \geq 0} \delta(p_1^- + p_2^- - p_3^- - s) \frac{|\lambda_1 - \lambda_2|^{2s}}{\Gamma(s+1)} \left(-\pi \frac{\Upsilon(p_1^+ + 1)\Upsilon(p_2^+ + 1)}{\Upsilon(p_3^+ + 1)} \right)^s, \end{aligned} \tag{5.19}$$

where $\Delta_i = -p_i^+ (p_i^- + 1/2)$ and $s = p_1^- + p_2^- - p_3^-$.

The δ-functions on the right-hand side enforce the conservation rules. The ρ, λ dependence is as expected from the Clebsh-Gordan coefficients; and the global conformal invariance dictates the z dependence. The only new piece of information is momentum-dependent coupling constants:

$$C^{p_1^+, p_2^+, p_3^+}_{p_1^-, p_2^-, p_3^-} = \frac{1}{s!} \left(-\pi \frac{\Upsilon(p_1^+ + 1)\Upsilon(p_2^+ + 1)}{\Upsilon(p_3^+ + 1)} \right)^s . \tag{5.20}$$

This result agrees with [6].

We should stress that the construction of the vertex operator and the computation of the correlation function is a priori valid only when $0 < p_i^+ < 1$. However the three point function is a holomorphic function with poles when p_3^+ is a nonzero integer.[14] We are going to see in the next section that for $p_i^+ > 1$ the three point function gives the correct amplitude associated with long string states.

Integral transform: we pause here to give some justification for the prescription eq. (5.12) which tells us that the correlation functions with insertion of more than one conjugate vertex operator is related by an integral transform to the correlation function with only one conjugate vertex operator. We expect the conjugate vertex operator $\widetilde{V}$ to be related to $V^\dagger$, the hermitean conjugate of V. Such an expectation is proved to be true due to the following identity for the three points function

$$\begin{aligned}&\left\langle V^{p_1^+,p_1^-}_{\bar\lambda_1\lambda_1}(z_1)V^{p_2^+,p_2^-}_{\bar\lambda_2\lambda_2}(z_2)\frac{\pi^{2p_3^-}\widetilde{V}^{p_3^+,p_3^-}_{\bar\lambda_3\lambda_3}(z_3)}{2^{2p_3^-}\Upsilon(p_3^++1)}\right\rangle = \\ &= -\pi^2\left\langle \frac{\pi^{2p_1^-}\widetilde{V}^{p_1^+,p_1^-}_{\bar\lambda_1\lambda_1}(z_1)}{2^{2p_1^-}\Upsilon(p_1^++1)}\frac{\pi^{2p_2^-}\widetilde{V}^{p_2^+,p_2^-}_{\bar\lambda_2\lambda_2}(z_2)}{2^{2p_2^-}\Upsilon(p_2^++1)}V^{p_3^+,p_3^-}_{\bar\lambda_3\lambda_3}(z_3)\right\rangle, \end{aligned} \tag{5.21}$$

which shows that

$$\widetilde{V}^{p^+,p^-} = -\frac{1}{\pi^2}\left(\frac{2}{\pi}\right)^{2p_1^-}\Upsilon(p_1^++1)(V^{p^+,p^-})^\dagger. \tag{5.22}$$

We first express $\langle \widetilde{V}_1\widetilde{V}_2V_3\rangle$, the right hand side of eq. (5.21), as an integral transform:

$$\mathcal{J} \equiv \frac{(p_2^+)^2}{\pi}\int d^2\rho_2\, e^{2p_2^+\lambda_2\rho_2}\, e^{2p_2^+\bar\lambda_2\bar\rho_2}\left\langle \widetilde{V}^{p_1^+,p_1^-}_{\bar\lambda_1\lambda_1}(0)V^{-p_2^+,-p_2^--1}_{\bar\rho_2\rho_2}(1)V^{p_3^+,p_3^-}_{\bar\lambda_3\lambda_3}(\infty)\right\rangle. \tag{5.23}$$

The p^+ conservation rule reads $p_1^+ + p_2^+ - p_3^+ = 0$. After the change of variables: $\eta = 2p_2^+(\lambda_3-\rho_2)(\lambda_2-\lambda_1)$ the integral transform, $\mathcal{J}$, reads

$$\begin{aligned}\mathcal{J} &= \frac{(p_2^+)^2}{\pi}\left|e^{2\lambda_3(2p_1^+\lambda_1+2p_2^+\lambda_2)}\right||\lambda_2-\lambda_1|^{2s}(2p_2^+)^{2s}\times \\ &\times\frac{1}{\Gamma(-s)}\left(-\pi\frac{\Upsilon(p_3^++1)\Upsilon(-p_2^++1)}{\Upsilon(p_1^++1)}\right)^{-s-1}\int\frac{d\eta d\bar\eta}{\pi}|e^{-\eta}\eta^{-s-1}|^2,\end{aligned} \tag{5.24}$$

where we have defined $s = p_1^- + p_2^- - p_3^-$. The η integral is evaluated to be[15] $(-1)^{s+1}\Gamma(-s)/\Gamma(s+1)$. Finally we use the identity $(p_2^+)^2/\Upsilon(-p_2^++1) = -\Upsilon(p_2^++1)$ to express

$$\begin{aligned}\mathcal{J} &= -\frac{1}{\pi^2}\frac{\left(\frac{2}{\pi}\right)^{2p_1^-}\Upsilon(p_1^++1)\left(\frac{2}{\pi}\right)^{2p_2^-}\Upsilon(p_2^++1)}{\left(\frac{2}{\pi}\right)^{2p_3^-}\Upsilon(p_3^++1)}\times \\ &\times\left|e^{2\lambda_3(2p_1^+\lambda_1+2p_2^+\lambda_2)}\right||\lambda_2-\lambda_1|^{2s}\frac{1}{\Gamma(s+1)}\left(-\pi\frac{\Upsilon(p_1^++1)\Upsilon(p_2^++1)}{\Upsilon(p_3^++1)}\right)^{s}.\end{aligned} \tag{5.25}$$

We recognize in the first line the normalisation factor which enters (eq. (5.21)) and in the second the correlator $\langle V_1V_2\widetilde{V}_3\rangle$. This proves our claim.

[14] $\Upsilon(1+p)$ admits a pole when $p=-n$, $n\geq 1$, and a zero when $p=n$, $n\geq 0$.

[15] See appendix C for details.

Normalization: given the symmetry in lightcone time reversal, i.e. $u \to -u$, it is natural to look for a normalized vertex operator, $\hat{V}$, and its conjugate, $\hat{V}^\dagger$, such that the three-point function are CPT invariant:

$$\langle \hat{V}_1 \hat{V}_2 \hat{V}_3^\dagger \rangle = \langle \hat{V}_1^\dagger \hat{V}_2^\dagger \hat{V}_3 \rangle \tag{5.26}$$

and the two-point amplitude satisfies

$$\langle \hat{V}^{p^+_, p^-} (\hat{V}^{-p^+_, -p^-})^\dagger \rangle = 1\,. \tag{5.27}$$

In light of (eq. (5.21)) we realize that

$$\hat{V}^{p^+,p^-}_{\bar{\lambda}\lambda} = \left(\frac{2}{\pi}\right)^{p^-} (-\Upsilon(p^+ + 1))^{\frac{1}{2}} e^{-2p^+\lambda\bar{\lambda}} V^{p^+,p^-}_{\bar{\lambda}\lambda}\,, \tag{5.28}$$

$$(\hat{V}^{p^+,p^-}_{\bar{\lambda}\lambda})^\dagger = \pi \left(\frac{\pi}{2}\right)^{p^-} (-\Upsilon(p^+ + 1))^{-\frac{1}{2}} e^{-2p^+\lambda\bar{\lambda}} \widetilde{V}^{p^+,p^-}_{\lambda\bar{\lambda}}\,. \tag{5.29}$$

Two-point function: the unit field is obtained from V^{p^+,p^-} in the limit where $p^+ = 0$, $p^- = 0$ [16]. We can therefore obtain the two point function by considering this limit in the three point function. Starting from the expression eq. (5.19) the limit is easily taken. Since $\Upsilon(p+1) \to -p$ as p approaches zero, only the term $s = 0$ survives. For the other factors entering the three-point function the limit can be trivially taken. One finds

$$\left\langle V^{p_1^+,p_1^-}_{\bar{\lambda}_1\lambda_1}(z_1) \widetilde{V}^{p_2^+,p_2^-}_{\bar{\rho}_2\rho_2}(z_2) \right\rangle = \frac{1}{\pi}|z_1 - z_2|^{-4\Delta} \delta(p_1^+ - p_2^+)\delta(p_1^- - p_2^-)|e^{2p_1^+\lambda_1\rho_2}|^2\,, \tag{5.30}$$

where $\Delta = -p_1^+(p_1^- + \frac{1}{2})$ as before. Using the normalized vertex operators, $\hat{V}$, we have indeed

$$\left\langle \hat{V}^{p_1^+,p_1^-}_{\bar{\lambda}_1\lambda_1}(0) (\hat{V}^{p_2^+,p_2^-}_{\bar{\rho}_2\rho_2})^\dagger(1) \right\rangle = \frac{1}{\pi}\delta(p_1^+ - p_2^+)\delta(p_1^- - p_2^-) e^{-2p_1^+|\lambda_1-\rho_2|^2}\,. \tag{5.31}$$

This completes the discussion on normalization of the vertex operators.

5.2 Four-point function: 3–1

The four-point function describing $3 \to 1$ scattering,

$$\left\langle V^{p_1^+,p_1^-}_{\bar{\lambda}_1\lambda_1}(z_1) V^{p_2^+,p_2^-}_{\bar{\lambda}_2\lambda_2}(z_2) V^{p_3^+,p_3^-}_{\bar{\lambda}_3\lambda_3}(z_3) \widetilde{V}^{p_4^+,p_4^-}_{\bar{\rho}_4\rho_4}(z_4) \right\rangle \tag{5.32}$$

is expressed in terms of the integral, $\mathcal{I}_{\vec{\lambda},\vec{p}^+}(\vec{z})$, with $p_4^+ = p_1^+ + p_2^+ + p_3^+$ and $p_4^+\lambda_4 = p_1^+\lambda_1 + p_2^+\lambda_2 + p_3^+\lambda_3$. Using the properties of $\mathcal{I}$ under the projective transformations of z we get

$$\mathcal{I}_{\vec{\lambda},\vec{p}^+}(\vec{z}) = \prod_{i<j} |z_i - z_j|^{p_i^+ + p_j^+} |z|^{-(p_1^+ + p_2^+)} |1 - z|^{-(p_2^+ + p_3^+)} I_{\vec{\lambda}}(z)\,, \tag{5.33}$$

[16] Note $\hat{V}$ blows up in this limit.

where z being the cross ratio $z = \dfrac{(z_2 - z_1)(z_3 - z_4)}{(z_2 - z_4)(z_3 - z_1)}$ and

$$\begin{aligned} I_{\vec\lambda}(z) &\equiv \int d^2w |\Delta_{\vec\lambda,z}(w)|^2 \\ &= \int d^2w \left| w^{p_1^+}(w-z)^{p_2^+}(w-1)^{p_3^+}\left(\frac{p_1^+\lambda_1}{w} + \frac{p_2^+\lambda_2}{w-z} + \frac{p_3^+\lambda_3}{w-1}\right)\right|^2 . \end{aligned} \tag{5.34}$$

According to the general rules for chiral splitting of integrals presented in appendix D, this integral can be written as

$$I_{\vec\lambda}(z) = \frac{\sin\pi p_1^+ \sin\pi p_2^+}{\sin\pi(p_1+p_2)}\left|\int_0^z |\Delta_{\vec\lambda,z}(x)|dx\right|^2 + \frac{\sin\pi p_3^+ \sin\pi p_4^+}{\sin\pi(p_1+p_2)}\left|\int_1^\infty |\Delta_{\vec\lambda,z}(x)|dx\right|^2 . \tag{5.35}$$

The line integrals can in turn be expressed in terms of hypergeometric functions

$$\int_0^z |\Delta_{\vec\lambda,z}(x)| = \frac{\Gamma(p_1^+ + 1)\Gamma(p_2^+ + 1)}{\Gamma(p_1^+ + p_2^+ + 1)} f_{\vec\lambda}(z) , \tag{5.36}$$

$$\int_1^\infty |\Delta_{\vec\lambda,z}(x)| = \frac{\Gamma(-p_4^+)\Gamma(p_3^+ + 1)}{\Gamma(-p_1^+ - p_2^+)} g_{\vec\lambda}; \tag{5.37}$$

with $f_{\vec\lambda}(z) = \sum_{i=1}^3 \lambda_i f_i(z)$, $g_{\vec\lambda} = \sum_{i=1}^3 \lambda_i g_i(z)$ where

$$\begin{aligned} f_1(z) &\equiv z^{p_1^+ + p_2^+} F\left(-p_3^+, p_1^+, p_1^+ + p_2^+ + 1; z\right) , \\ f_2(z) &\equiv -z^{p_1^+ + p_2^+} F\left(-p_3^+, p_1^+ + 1, p_1^+ + p_2^+ + 1; z\right) , \\ f_3(z) &\equiv \frac{z^{p_1^+ + p_2^+ + 1} p_3^+}{p_1^+ + p_2^+ + 1} F\left(1 - p_3^+, p_1^+ + 1, p_1^+ + p_2^+ + 2; z\right) , \end{aligned} \tag{5.38}$$

$$\begin{aligned} g_1(z) &\equiv \frac{-p_1^+}{p_1^+ + p_2^+} F\left(-p_2^+, -p_4^+, -p_1^+ - p_2^+ + 1; z\right) , \\ g_2(z) &\equiv \frac{-p_2^+}{p_1^+ + p_2^+} F\left(1 - p_2^+, -p_4^+, -p_1^+ - p_2^+ + 1; z\right) , \\ g_3(z) &\equiv F\left(-p_2^+, -p_4^+, -p_1^+ - p_2^+; z\right) . \end{aligned} \tag{5.39}$$

Note that these functions satisfy the Gauss recursions identities

$$\sum_{i=1}^3 g_i(z) = \sum_{i=1}^3 f_i(z) = 0 . \tag{5.40}$$

Overall this gives

$$I_{\vec\lambda}(z) = -\pi\left(\frac{\Upsilon(p_1^+ + 1)\Upsilon(p_2^+ + 1)}{\Upsilon(p_1^+ + p_2^+ + 1)}|f_{\vec\lambda,\vec p}(z)|^2 + \frac{\Upsilon(p_1^+ + p_2^+ + 1)\Upsilon(p_3^+ + 1)}{\Upsilon(p_4^+ + 1)}|g_{\vec\lambda,\vec p}(z)|^2\right) . \tag{5.41}$$

The total amplitude is then given by

$$\left\langle V^{p_1^+,p_1^-}_{\bar\lambda_1\lambda_1}(0) V^{p_2^+,p_2^-}_{\bar\lambda_2\lambda_2}(z) V^{p_3^+,p_3^-}_{\bar\lambda_3\lambda_3}(1) \widetilde V^{p_4^+,p_4^-}_{\bar\rho_4\rho_4}(\infty)\right\rangle = e^{\rho_4(2p_1^+\lambda_1 + 2p_2^+\lambda_2 + 2p_3^+\lambda_3)} \frac{(I_{\vec\lambda}(z))^s}{\Gamma(s+1)} \tag{5.42}$$

with $s = p_1^- + p_2^- + p_3^- - p_4^-$ being an integer[17] and $p_1^+ + p_2^+ + p_3^+ = p_4^+$. The four point function can now be expressed as a linear combination of conformal blocks

$$\frac{(I_{\vec{\lambda}}(z))^s}{\Gamma(s+1)} = \sum_{\eta=0}^{s} C^{p_1^+,p_2^+,p_1^++p_2^+}_{p_1^-,p_2^-,p_1^-+p_2^--\eta} C^{p_1^++p_2^+,p_3^+,p_4^+}_{p_1^-+p_2^--\eta,p_3^-,p_4^-} |f_{\vec{\lambda},\vec{p}}(z)|^{2s} | g_{\vec{\lambda},\vec{p}}(z)|^{2(s-\eta)} , \tag{5.43}$$

where C_{p_1,p_2,p_3} is the three-particle coupling constant eq. (5.20).

The factorization of the four point function shows that if we start with short string vertex operators having $0 < p_i^+ < 1$ there will be no intermediate states corresponding to long strings since $p_1^+ + p_2^+ + p_3^+ = p_4^+ < 1$. In order to have long string states propagating in the intermediate channel we either have to insert external long string states $p_i^+ > 1$ *or* insert external state with, for instance $0 > p_2^+ > -1$, by our integral transform rule this correspond to inserting one more conjugate short string vertex operator.

5.3 Four-point function: 2–2

We now want to construct the amplitude with two conjugate fields, such an amplitude is related by an integral transform to the previous amplitude

$$\begin{aligned}&\left\langle V^{p_1^+,p_1^-}_{\bar{\lambda}_1\lambda_1}(0)\widetilde{V}^{p_2^+,p_2^-}_{\bar{\rho}_2\rho_2}(z)V^{p_3^+,p_3^-}_{\bar{\lambda}_3\lambda_3}(1)\widetilde{V}^{p_4^+,p_4^-}_{\bar{\rho}_4\rho_4}(\infty)\right\rangle \\ &\qquad = e^{\rho_4(2p_1^+\lambda_1+2p_3^+\lambda_3)}\int d^2\lambda_2 \left|e^{2p_2^+\lambda_2(\rho_2-\rho_4)}\right|^2 \frac{(I_{\vec{\lambda}}(z))^s}{\Gamma(s+1)} ,\end{aligned} \tag{5.44}$$

where $I_{\vec{\lambda}}(z)$ is defined as in the previous section except that we have to change p_2^+ into $-p_2^+$ in all the expressions and $s = p_1^- - p_2^- + p_3^- - p_4^- - 1$. We have:

$$I_{\vec{\lambda}} = C_{12}|f_{\vec{\lambda}}|^2 + C_{34}|g_{\vec{\lambda}}|^2 , \tag{5.45}$$

where

$$C_{12} = \frac{\Upsilon(p_1^+ + 1)\Upsilon(-p_2^+ + 1)}{\Upsilon(p_1^+ - p_2^+ + 1)} , \tag{5.46}$$

$$C_{34} = \frac{\Upsilon(-p_4^+)\Upsilon(p_3^+ + 1)}{\Upsilon(p_2^+ - p_1^+)} . \tag{5.47}$$

For convenience we introduce the shorthand notations:

$$A = C_{12}|f_2|^2 + C_{34}|g_2|^2 , \tag{5.48}$$

$$B = (\lambda_1 - \lambda_3)(C_{12}f_2f_1 + C_{34}g_2g_1) - \lambda_3 A , \tag{5.49}$$

$$AC - B\bar{B} = C_{12}C_{34}|\lambda_1 - \lambda_3|^2|f_2g_1 - g_2f_1|^2 , \tag{5.50}$$

to write $I_{\vec{\lambda}} = A\lambda_2\bar{\lambda}_2 + \bar{B}\lambda_2 + B\bar{\lambda}_2 + C$.

[17] Due to the presence of the gamma function in the denominator the amplitude is zero if s is not a positive integer

In appendix C we have computed such an integral transform. Using the result of (eq. (C.6)) we obtain the following expression for the four-point function with 2 conjugate fields

$$e^{\rho_4(2p_1^+\lambda_1+2p_3^+\lambda_3)}\left|e^{-2p_2^+(\rho_2-\rho_4)\frac{B}{A}}\right|^2\left(\frac{D}{-4p_2^+|\rho_2-\rho_4|^2}\right)^{s+1}\frac{1}{A}I_{-s-1}\left(\frac{D}{A}\right), \tag{5.51}$$

where we have denoted $D\equiv 2p_2^+|\rho_2-\rho_4|\sqrt{B\bar{B}-AC}$ and I_{-s-1} the modified Bessel function of the first kind. We can evaluate this quantity to be

$$D=\sqrt{-C_{12}C_{34}}\,\left|2p_2^+(\rho_2-\rho_4)(\lambda_1-\lambda_3)z^{p_1^+-p_2^+}(1-z)^{p_3^+-p_2^+}\right|. \tag{5.52}$$

In order to show this we first check that

$$\begin{aligned}\frac{p_2^+}{z(1-z)}f_1(z)&=a(z)^{-1}\partial_z(a(z)f_2(z))\,,\\ \frac{p_2^+}{z(1-z)}g_1(z)&=a(z)^{-1}\partial_z(a(z)g_2(z))\,,\end{aligned} \tag{5.53}$$

where $a(z)=z^{-p_1^+}(1-z)^{p_1^+-p_4^+}$. This shows that $AC-B\bar{B}$ is just a Wronskien which can be evaluated to be

$$AC-B\bar{B}=C_{12}C_{34}|(\lambda_1-\lambda_3)z^{p_1^+-p_2^+}(1-z)^{p_3^+-p_2^+}|^2\,. \tag{5.54}$$

The expression (eq. (5.51)) of the correlation function is similar to the one obtained in [6]: Up to a p^+ dependent proportionality factor it has the same z dependence. The factorisation of this amplitude performed in [6] can thus be applied to our case and indeed shows that long strings propagate as intermediate states.

6. Ward identities and flat space limit

In this section we shall check that the N-point function we proposed satisfies all the required Ward identities. Our amplitudes also coincide with those in the flat space upon sending H to zero. Let us first denote the chiral part of the correlation function as

$$G_{\lambda,C}(\vec{z})\equiv\left(\prod_{i\neq j}(z_i-z_j)^{-p_i^+p_j^-}\right)\left(\oint_C dw\Delta_{\vec{\lambda},\vec{p}^+}(w,\vec{z})\right)^s\,, \tag{6.1}$$

where C is a contour of integration.

6.1 Conformal ward identities

We can easily compute the transformation rule of $G_{\lambda,C}(\vec{z})$ and our n-point function (eq. (5.10)) under the projective transformations

$$z\to\tilde{z}=\frac{(az+b)}{(cz+d)}\,. \tag{6.2}$$

Under such transformation $w - z_i$ becomes $(w - z_i)/(cw + d)(cz_i + d)$ and dw becomes $dw/(cw + d)^2$. Therefore the factor $\prod_{i\neq j}(z_i - z_j)^{-p_i^+ p_j^- - p_j^+ p_i^-}$ is transformed as

$$\prod_i (cz_i + d)^{p_i^+(\sum_{j\neq i} p_j^-) + (\sum_{j\neq i} p_j^+) p_i^-} \prod_{i\neq j} (z_i - z_j)^{-p_i^+ p_j^-} \tag{6.3}$$

using the conservation rules for the momenta eq. (5.3) this gives

$$\prod_i (cz_i + d)^{-2p_i^+ p_i^- + p_i^+(s-1)} \prod_{i\neq j} (z_i - z_j)^{-p_i^+ p_j^-} . \tag{6.4}$$

The integral $\oint_C dw \Delta_{\vec{\lambda},\vec{p}^+}(w, \vec{z})$ under a projective transformation becomes

$$\prod_i (cz_i + d)^{-p_i^+} \oint_C dw \prod_{i=1}^n (w - z_i)^{p_i^+} \left(\sum_{i=1}^n \frac{p_i^+ \lambda_i (cz_i + d)}{(w - z_i)(cw + d)} \right). \tag{6.5}$$

One uses the identity

$$\frac{(cz_i + d)}{(w - z_i)(cw + d)} = \frac{1}{w - z_i} - \frac{c}{cw + d} \tag{6.6}$$

and the conservation rules $\sum_i p_i^+ \lambda_i = 0$ to see that the integrand is exactly $dw \Delta_{\vec{\lambda},\vec{p}^+}(w, \vec{z})$. Overall one gets

$$G_{\lambda,C}\left(\frac{a\vec{z} + b}{c\vec{z} + d}\right) = \prod_{i=1}^n (cz_i + d)^{-2p_i^+ p_i^- - p_i^+} G_{\lambda,C}(\vec{z}). \tag{6.7}$$

which shows that the conformal weight of the vertex operator $V_\lambda^{p^\pm}$ is given by

$$-p^+ \left(p^- + \frac{1}{2}\right) \tag{6.8}$$

as expected.

6.2 Algebraic ward identities

For each generator of the Lie algebra $J^a = T, J, J^+, J^-$, we expect a conservation rule $(\sum_i J_i^a)\Gamma_{\vec{\lambda},\vec{\bar{\lambda}},s} = 0$. The T conservation rule is the conservation of light-cone momenta $\sum_i p_i^+ = 0$. The J^- conservation rule is equivalent to $\sum_i p_i^+ \lambda_i = 0$. The J^+ conservation rule amounts to invariance under translation in λ, $\Gamma_{\vec{\lambda},\vec{\bar{\lambda}},s} = \Gamma_{\vec{\lambda}+a,\vec{\bar{\lambda}},s}$. This is clear first since $\sum_i p_i^+ a = 0$ and also since

$$\prod_{i=1}^n (w - z_i)^{p_i^+} \left(\sum_{i=1}^n \frac{p_i^+ a}{(w - z_i)} \right) = a \frac{\partial}{\partial w} \left(\prod_{i=1}^n (w - z_i)^{p_i^+} \right). \tag{6.9}$$

The J conservation rule comes from the fact that Γ is homogeneous of degree $s = \sum_i p_i^-$,

$$\Gamma_{a\vec{\lambda},\vec{\bar{\lambda}},s} = a^{\sum_i p_i^-} \Gamma_{\vec{\lambda},\vec{\bar{\lambda}},s} . \tag{6.10}$$

This is clear since $\delta(\sum_i p_i^+ \lambda_i)$ is homogeneous of degree -1 and $\Delta_{\vec{\lambda},\vec{p}^+}(w, \vec{z})$ is homogenous of degree $+1$ in λ.

6.3 Knizhnik-Zamolodchikov equation

Let us denote by $\hat{C}_{ij} = J_i T_j + T_i J_j + \frac{1}{2} J_i^+ J_j^- + \frac{1}{2} J_i^- J_j^+$, the Casimir operator acting non-trivially on the i and j factor of $\otimes_{i=1}^n V^{p_i^+, p_i^-}$. In the coherent state basis this operator can be represented as a differential operator

$$\hat{C}_{ij} = -p_i^+ p_j^- - p_i^- p_j^+ - (\lambda_j - \lambda_i)(p_i^+ \partial_{\lambda_j} - p_j^+ \partial_{\lambda_i}), \qquad i \neq j\,. \tag{6.11}$$

We are going to prove that $G_{\lambda,C}(\vec{z})$ satisfies the KZ equation[18]

$$\frac{\partial}{\partial z_i} G_{\lambda,C}(\vec{z}) = \sum_{j \neq i} \frac{\hat{C}_{ij}}{z_i - z_j} G_{\lambda,C}(\vec{z})\,. \tag{6.12}$$

First it is clear that

$$\frac{\partial}{\partial z_i} \prod_{i \neq j} (z_i - z_j)^{-p_i^+ p_j^-} = -\sum_{j \neq i} \frac{p_i^+ p_j^- + p_i^- p_j^+}{z_i - z_j} \prod_{i \neq j} (z_i - z_j)^{-p_i^+ p_j^-}\,. \tag{6.13}$$

Then $\partial_{z_i} \oint_C dw \Delta_{\vec{\lambda},\vec{p}^+}(w, \vec{z})$ is given by

$$\oint_C dw \prod_j (w - z_j)^{p_j^+} \frac{p_i^+}{z_i - w} \left(-\sum_j \frac{p_j^+ \lambda_j}{z_j - w} + \frac{\lambda_i}{z_i - w} \right). \tag{6.14}$$

Moreover,

$$0 = \oint_C d \left(\prod_j (w - z_i)^{p_j^+} \frac{p_i^+ \lambda_i}{z_i - w} \right) = \oint_C dw \prod_j (w - z_i)^{p_j^+} \frac{p_i^+}{z_i - w} \left(\sum_j \frac{p_j^+ \lambda_i}{z_j - w} - \frac{\lambda_i}{z_i - w} \right). \tag{6.15}$$

Summing the two identities one gets

$$\begin{aligned}
\frac{\partial}{\partial z_i} G_{\lambda,C}(\vec{z}) &= \oint_C dw \prod_j (w - z_i)^{p_j^+} \sum_{j \neq i} \frac{\lambda_i - \lambda_j}{z_i - z_j} \left(\frac{1}{z_j - w} - \frac{1}{z_i - w} \right) \\
&= \sum_{j \neq i} \frac{\lambda_i - \lambda_j}{z_i - z_j} (p_i^+ \partial_{\lambda_j} - p_j^+ \partial_{\lambda_i}) \oint_C dw \Delta_{\vec{\lambda},\vec{p}^+}(w, \vec{z})\,.
\end{aligned} \tag{6.16}$$

The KZ equation now follows trivially from eq. (6.13) and eq. (6.16).

6.4 Flat space limit

There are several unusual features in the amplitude eq. (5.10) as we have written it down which obscure comparison with flat space. The dependence on H has been made implicit, however flat space is obtained precisely as H goes to 0. In particular, what we have implicitly done is to rescale light-cone momenta according to

$$p^+ \to H p^+\,, \qquad p^- \to \frac{p^-}{H}\,.$$

[18] This is true without using any conservation rule!

One feature of the amplitude is presence of shifts in the conservation rule of p^-, the shifts are proportional to H and they go away when H is taken to 0. but the summation over s remains. The nontrivial issue is due to the remaining integral over w in eq. (5.11). When we reinstate H as above, the p_i^+ in the integral get rescaled to zero. Thus the integral reduces simply to

$$\int d^2w \left(\sum_{i=1}^{n} \frac{p_i^+ \lambda_i}{w - z_i} \right) \left(\sum_{j=1}^{n} \frac{p_j^+ \bar{\lambda}_j}{\bar{w} - \bar{z}_j} \right). \tag{6.17}$$

There are two types of terms appearing when we open the bracket, depending on whether we combine same or different terms ($i = j$ or $i \neq j$):

$$\int \frac{d^2w}{|w|^2} \qquad \text{and} \qquad \int \frac{d^2w}{(w - z_i)(\bar{w} - \bar{z}_j)}.$$

After properly regularizing, the first one is set to zero, while the second one is taken to be $-\ln(\mu|z_i - z_j|)$. Thus the summation over s in eq. (5.10) gives an exponential

$$\exp(-\sum_{i \neq j} p_i^+ p_j^+ \lambda_i \bar{\lambda}_j \ln(\mu|z_i - z_j|)) = \prod_{i \neq j} |z_i - z_j|^{\frac{-p_\perp^i p_\perp^j}{4}},$$

if we also set $p_\perp = 2p^+ \lambda$.

Then by the usual arguments the final result for the physical amplitude is shown to be independent of the infrared cutoff μ as long as momentum conservation is observed.

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A. Light cone treatment

We would now go to the light-cone gauge so that the model becomes soluble in this nontrivial background. The light cone quantization of this model was first studied by [16, 17]. The key is that the cubic interaction term becomes quadratic when one imposes $u = X^+ = p^+\tau$. In a generic curved background the light cone gauge cannot be imposed. However it has been shown by Horowitz and Steif [19] that the light-cone choice is consistent provided that one has a light-like Killing vector, as it is the case of the plane wave background.

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We generalize the standard treatment [20] of light-cone gauge fixing and the construction of the light-cone hamiltonian to the case where the NS field is present.

The action is

$$\mathcal{L} = \frac{1}{2}\int d^2\sigma \; \partial_a X^m \, \partial_\beta X^n \left(\sqrt{h}\, h^{a\beta} G_{mn} + \epsilon^{a\beta} B_{mn}\right), \tag{A.1}$$

where

$$h = h_{\tau\sigma}\, h_{\tau\sigma} - h_{\tau\tau}\, h_{\sigma\sigma}\,. \tag{A.2}$$

The generalized momenta are

$$\Pi_m = \frac{\delta \mathcal{L}}{\delta \dot{X}^m} = \sqrt{h}\, G_{mn}(h^{\tau\tau}\dot{X}^n + h^{\tau\sigma}\acute{X}^n) + B_{mn}\acute{X}^n\,. \tag{A.3}$$

Now solve for $\dot{X}$

$$\dot{X}^m = \frac{G^{mn}}{\sqrt{h}\, h^{\tau\tau}}\left(\Pi_n - B_{np}\acute{X}^p\right) - \frac{h^{\tau\sigma}}{h^{\tau\tau}}\,\acute{X}^m\,. \tag{A.4}$$

Eliminating $\dot{X}$ from the action, after a tedious computation, a simple answer emerges:

$$\mathcal{L} = \frac{1}{2}\int d^2\sigma \, \frac{1}{\sqrt{h}h^{\tau\tau}}\left(\Pi^2 - \acute{X}^2 + \acute{X}^m B_{mn} G^{np} B_{pq}\acute{X}^q\right). \tag{A.5}$$

Note that we have made use of eq. (A.2) to eliminate $h^{\sigma\sigma}$. We then Legendre-transform to arrive at the hamiltonian

$$\begin{aligned}
\mathcal{H} &= \Pi_m \dot{X}^m - \mathcal{L} \\
&= \Pi_m\left(\frac{G^{mn}}{\sqrt{h}h^{\tau\tau}}(\Pi_n - B_{np}\acute{X}^p) - \frac{h^{\tau\sigma}}{h^{\tau\tau}}\acute{X}^m\right) - \mathcal{L} \\
&= \frac{1}{2\sqrt{h}\, h^{\tau\tau}}\left[\Pi^2 + \acute{X}^2 - 2\,\Pi_m G^{mn} B_{np}\acute{X}^p - 2\acute{X}^m B_{mn} G^{np} B_{pq}\acute{X}^q\right] - \frac{h^{\tau\sigma}}{h^{\tau\tau}}\,\Pi\cdot\acute{X}
\end{aligned} \tag{A.6}$$

The independent components of the worldsheet metric above play the role of lagrangian multipliers thus a variation with respect to them gives the contraint equations.

$$\begin{aligned}
\Pi^2 + \acute{X}^2 - 2\,\Pi^m\, B_{mp}\,\acute{X}^p - 2\acute{X}^m B_{mn} G^{np} B_{pq}\acute{X}^q &= 0 \\
\Pi\cdot\acute{X} &= 0\,.
\end{aligned} \tag{A.7}$$

The first allows us to obtain the light-cone hamiltonian, while the second expresses the longitudinal coordinate in terms of transverse physical degrees of freedom. Light-cone gauge is selected by setting $X^+ = 2\pi a' p^+ \tau$ and $\Pi_- = 2\pi a' p^+$.

We now specialize to the case of the Nappi-Witten spacetime with the metric and NS B-field pertinent to our model, we get the light-cone constraints

$$2\,p^+\Pi_+ + \Pi_1^2 + \Pi_2^2 + 2\,\mu\,(a_1\Pi_2 - a_2\Pi_1) + \acute{a}_1^2 + \acute{a}_2^2 + 2\,\mu\,(\acute{a}_1 a_2 - \acute{a}_2 a_1) + \mu^2\,(a_1^2 + a_2^2) = 0 \tag{A.8}$$

$$\acute{a}_1\Pi_1 + \acute{a}_2\Pi_2 + \acute{v} = 0\,, \tag{A.9}$$

where $\mu = \frac{p^+ H}{2}$.

The light-cone hamiltonian, $\mathcal{H}_{lc}$, is identified with one of the light-cone momenta:

$$\begin{aligned}\mathcal{H}_{lc} &\equiv -p^+\Pi_+ = p^+\Pi^- \qquad\qquad (A.10)\\ 2\,\mathcal{H}_{lc} &= \Pi_1^2+\Pi_2^2+2\,\mu\,(a_1\Pi_2-a_2\Pi_1)+\acute{a}_1^2+\acute{a}_2^2+2\,\mu\,(\acute{a}_1a_2-\acute{a}_2a_1)+\mu^2\,(a_1^2+a_2^2)\,.\end{aligned}$$

And we can also solve for the longitudinal coordinate $\acute{v}$ in terms of the dynamical transverse physical fields:

$$\begin{aligned}\acute{v} &= -(\acute{a}_1\Pi_1+\acute{a}_2\Pi_2)\\ &= -\dot{a}_1a_1'-\dot{a}_2a_2'-\mu\,(a_2a_1'-a_1a_2')\,.\end{aligned} \qquad (A.11)$$

The form of this last expression cannot be naively guessed and differs from the flat-space result by the H-dependent terms. This condition, integrated over σ, produces the left-right matching constraint on the physical Hilbert space.

Let us perform a Legendre transformation. The light-cone action takes the form:

$$\mathcal{L}=\frac{1}{2}(\dot{a}_1^2+\dot{a}_2^2)-\frac{1}{2}(\acute{a}_1^2+\acute{a}_2^2)-\mu\,(a_1\dot{a}_2-a_2\dot{a}_1)+\mu\,(a_1\acute{a}_2-a_2\acute{a}_1)\,. \qquad (A.12)$$

This is naively equivalent to what one gets by setting $u=X^+=p^+\tau$ in the conformal gauge action; but the subtlety is that in generic curved backgrounds the conformal gauge is incompatible with the light-cone gauge choice. For the pp-wave backgrounds this is assured due to existence of null Killing vector which can be taken as definition of global time[19].

A.1 Solving the equations of motion

The equations of motion of the closed string are, in worldsheet light-cone coordinates $\sigma^\pm=\tau\pm\sigma$,

$$\partial_+\partial_-a^i+\mu\,\epsilon_{ij}\,\partial_-a^j=0\,. \qquad (A.13)$$

The solutions are

$$a=\frac{(a^1+i\,a^2)}{2}=i\,e^{i\mu\sigma^+}\left(\sum_{n\in\mathbb{Z}}\frac{\tilde{a}_n}{n+\mu}\,e^{-i(n+\mu)\sigma^+}+\frac{a_n}{n-\mu}\,e^{-i(n-\mu)\sigma^-}\right) \qquad (A.14)$$

$$\bar{a}=\frac{(a^1-i\,a^2)}{2}=i\,e^{-i\mu\sigma^+}\left(\sum_{n\in\mathbb{Z}}\frac{\bar{\tilde{a}}_n}{n-\mu}\,e^{-i(n-\mu)\sigma^+}+\frac{\bar{a}_n}{n+\mu}\,e^{-i(n+\mu)\sigma^-}\right). \qquad (A.15)$$

With the introduction of a time-dependent twisting, these can in turn be written in terms of free fields, X, satisfying $\partial_+\partial_-X=0$, $X(\sigma+2\pi,\tau)=e^{-i2\pi\mu}X(\sigma,\tau)$.:

$$a=e^{i\mu\sigma^+}X\,;\qquad \bar{a}=e^{-i\mu\sigma^+}\bar{X}\,. \qquad (A.16)$$

The free fields X are orbifold fields, but the physical fields must remain periodic. The corresponding lagrangian in X is the standard free string action. This is not contradictory to the original interacting theory because the field redefinitions are time-dependent. The choice of the normalization factors in the mode expansion will be apparent in the next section.

The $\tilde{a}_0$ ($\bar{\tilde{a}}_0$) is the center-of-mass coordinates and should be identified with ρ ($\bar{\rho}$). Similarly we should identify a_0 ($\bar{a}_0$) with the radius, λ ($\bar{\lambda}$), of the classical trajectory. For non-zero H, particles on Nappi-Witten space move in circles, however position and radius of those are arbitrary. One can also see that the terms linear in τ are not allowed and thus there is no zero-mode momentum operators in the mode expansion above. If one takes the limit $H \to 0$, the frequency of the a_0 mode goes to zero and becomes the momentum operator of the limiting flat space.

A.2 Quantization

After solving the classical equations of motion we now proceed to quantize the system. We will find that the zero mode operators become noncommutative in the background of constant Neveu-Schwarz flux.

We first compute the (complex) momenta conjugate to a and $\bar{a}$

$$\Pi = \dot{a} - i\,\mu\, a\,, \qquad \bar{\Pi} = \dot{\bar{a}} + i\,\mu\,\bar{a}\,.$$

From the oscillator expansion of a eq. (A.14) we have

$$\Pi = e^{i\mu\sigma^+}\left[\sum_{n\in\mathbb{Z}} \tilde{a}_n\, e^{-i(n+\mu)\sigma^+} + a_n\, e^{-i(n-\mu)\sigma^-}\right]. \tag{A.17}$$

The Poisson bracket

$$\{\Pi(\sigma), \bar{a}(\sigma')\} = \frac{1}{2\pi}\delta(\sigma - \sigma') \tag{A.18}$$

implies

$$\begin{aligned} i\{a_n, \bar{a}_m\} &= \frac{1}{2}\delta_{n,-m}(m+\mu) \\ i\{\tilde{a}_n, \bar{\tilde{a}}_m\} &= \frac{1}{2}\delta_{n,-m}(m-\mu)\,. \end{aligned} \tag{A.19}$$

Replacing the Poisson brackets with commutators, we finally obtain:

$$\begin{aligned} [a_n, \bar{a}_m] &= \frac{1}{2}\delta_{n,-m}(n-\mu), \\ [\tilde{a}_n, \bar{\tilde{a}}_m] &= \frac{1}{2}\delta_{n,-m}(n+\mu)\,. \end{aligned} \tag{A.20}$$

The reality conditions are given by

$$a_n^\dagger = \bar{a}_{-n}\,; \qquad \tilde{a}_n^\dagger = \bar{\tilde{a}}_{-n}\,. \tag{A.21}$$

Recall that we have been working with complexified coordinate a and $\bar{a}$. We now would like to undo this and write $a_n = a_n^1 + i\,a_n^2$. The commutation relations can be safely reproduced by setting

$$\begin{aligned} \left[a_n^{1,2}, a_m^{1,2}\right] &= n\,\delta_{n,-m} & \left[\tilde{a}_n^{1,2}, \tilde{a}_m^{1,2}\right] &= n\,\delta_{n,-m} \\ \left[a_n^1, a_m^2\right] &= -i\,\mu\,\delta_{n,-m} & \left[a_n^1, a_m^2\right] &= i\,\mu\,\delta_{n,-m}\,. \end{aligned} \tag{A.22}$$

Unlike in flat space, the same formula holds for the zero mode. We are forced to make the highly surprising conclusion that the two directions are noncommutative already at the closed string level:

$$[a_0^1, a_0^2] = i\,\mu\,.$$

In terms of the complexified coordinates this means that a_0 and $\bar{\tilde{a}}_0$ are creation operators.

When the value of μ is restricted to be $0 \le \mu < 1$ creation operators are those with negative indices, $m < 0$. Unlike the case in flat space case, the right and left moving zero-modes here are not degenerate, i.e. they have frequencies of $+\mu$ and $-\mu$ respectively. This is reflected in the mode expansion already — the right and left moving zero-modes are independent of each other.

When $\mu = N + \epsilon$ where N is a positive integer the vacuum is annihilated by a_{N+m}, $\bar{\tilde{a}}_{N+m}$, $m > 0$ and $\tilde{a}_{-N+m}, \bar{a}_{-N+m}$, $m \ge 0$. The "zero modes" are given by $a_N, \tilde{a}_{-N}$ and the corresponding classical classical solution is

$$\frac{ie^{iN\sigma^+}}{\epsilon}\left(\tilde{a}_{-N} - a_N e^{2i\epsilon\tau}\right). \tag{A.23}$$

When $N = 0$ this is the geodesic motion, centering at $\tilde{a}_0/\mu$ and with a radius a_0/μ, oscillating in time at frequency μ. But for $N \neq 0$ this describes the motion of a "long string",[19]

$$e^{iN\sigma}\left(\frac{\tilde{a}_{-N}}{\epsilon}e^{iN\tau} - \frac{a_N}{\epsilon}e^{iN\tau}e^{2i\epsilon\tau}\right) \tag{A.24}$$

i.e. a 2-dimensional surface winding N times around the origin. It envelopes the geodesic — centered at $\tilde{a}_{-N}/\epsilon$ and with a radius of a_N/ϵ — and oscillates with a slow frequency ϵ. What we are witnessing here is a dynamical dielectric effect [21] such that the light-cone momentum is transmuted into winding number under the influence of the H field: For every increase of the light-cone momentum by unit value of $\frac{1}{2\pi\alpha'^2}H^{-1}$ the winding number of the ground state also increases by one.

The quantum hamiltonian can be obtained by substituting the string mode expansions into the classical expression and normal-ordering:

$$\frac{1}{2}p^+\mathcal{H}_{lc} = \sum_{n\in\mathbb{Z}}\left(:a_n\bar{a}_{-n}:\ \frac{n}{n-\mu} + :\tilde{a}_n\bar{\tilde{a}}_{-n}:\ \frac{n+2\mu}{n+\mu}\right) + \mu(1-\mu) - \frac{1}{12}\,. \tag{A.25}$$

The normal ordering constant is given by the identity $\sum_{n>0}(n+\mu) = -1/12 + \mu(1-\mu)/2$. One could write the hamiltonian in a more compact way by introducing the properly normalized number operators, $N_n = \frac{a_n\bar{a}_{-n}}{n-\mu}$, $\tilde{N}_n = \frac{\tilde{a}_n\bar{a}_{-n}}{n+\mu}$. The total left and right occupation numbers are given by $N = \sum_n nN_n$, $\tilde{N} = \sum n\tilde{N}_n$. The transverse angular momenta are defined to be $J = \sum_n N_n$, $\tilde{J} = \sum_n \tilde{N}_n$. The light cone hamiltonian is then given, up to the normal ordering constant, by

$$\frac{1}{2}p^+\mathcal{H}_{lc} = N + \tilde{N} + 2\mu J\,, \tag{A.26}$$

[19] See [8, 22] for a general definition and [11, 6] for an application of this notion to the Nappi-Witten model.

in agreement with Russo-Tseytlin [4]. The left-right matching constraint (eq. (A.11)), becomes formally equivalent to the flat space condition:

$$N = \tilde{N}\,.$$

B. Nappi-Witten algebra, representations and wave functions

The Nappi-Witten Lie algebra is a central extension of the two-dimensional Poincare algebra. The anti-hermitean generators of this algebra are denoted J, T, J^1, J^2.

$$[J^+, J^-] = 2\,i\,T \qquad [J, J^+] = i\,J^+ \qquad [J, J^-] = -i\,J^- \qquad [T, \cdot] = 0\,, \tag{B.1}$$

where $J^\pm = J^1 \mp i\,J^2$. The invariant metric is taken to be

$$\mathrm{tr}(J\,T) = 1\,, \qquad \mathrm{tr}(J^+ J^-) = 2\,, \tag{B.2}$$

which is of signature $- + ++$. This algebra possesses two Casimirs given by T and $\mathcal{C} = J^+J^-/2 + J^-J^+/2 + 2\,J\,T$. It will be important to note that this algebra possesses a linear automorphism $\mathcal{P}$, first discussed in [15], which acts as a charge conjugation:

$$\mathcal{P}(J^+) = J^-\,, \qquad \mathcal{P}(J^-) = J^+\,, \qquad \mathcal{P}(T) = -T\,, \qquad \mathcal{P}(J) = -J\,. \tag{B.3}$$

A general group element in the Nappi-Witten group can be parametrized by the coordinates a_1, a_2, u, v

$$g(a, u, v) = e^{HaJ_+ + H\bar{a}J_-} e^{HuJ + HvT}\,, \tag{B.4}$$

where $a = (a_1 + i\,a_2)/2$. Note that in terms of the corotating frame coordinates $x = e^{-iHu/2}a$ the group element can be written

$$g(a, u, v) = e^{\frac{HuJ+HvT}{2}} e^{HxJ_+ + H\bar{x}J_-} e^{\frac{HuJ+HvT}{2}}\,. \tag{B.5}$$

The metric on the group is given by

$$ds^2 = \frac{1}{H^2}\mathrm{tr}(g^{-1}dg)^2 = 2\,du\,dv + da_1^2 + da_2^2 - H(a_1\,da_2 - a_2\,da_1)\,du\,. \tag{B.6}$$

From the definition of $g(a, u, v)$ eq. (B.5) we can compute the product of two group elements

$$g_L\,g_R = g(a_L + e^{iHu_L}a_R, u_L + u_R, v_L + v_R + iH\,(a_L\bar{a}_R e^{-iHu_L} - a_R\bar{a}_L e^{iHu_L}))\,, \tag{B.7}$$

where $g_L = g(a_L, u_L, v_L)$ and $g_R = g(a_R, u_R, v_R)$. We can also compute the inverse

$$g^{-1}(a, u, v) = g(-ae^{-iHu}, -u, -v). \tag{B.8}$$

Overall this gives

$$g_L^{-1} g_R = g(e^{-iHu_L}(a_R - a_L), u_R - u_L, v_R - v_L - iH(a_L\bar{a}_R - a_R\bar{a}_L))\,. \tag{B.9}$$

The charge conjugation is acting as a parity transformation on the space-time

$$\mathcal{P}(g(a, u, v)) = g(\bar{a}, -u, -v)\,. \tag{B.10}$$

The metric (B.6) is invariant under the isometry group $G_L \times G_R$, $g \to g_L^{-1} g g_R$. Since the generator T is commuting this isometry group is seven dimensional. When we write the infinitesimal action of this group in terms of the coordinates one gets

$$
\begin{aligned}
T_L &= -\partial_v \,, & J_R^v &= \partial_v \,, \\
J_L &= -(\partial_u + i(a\partial_a - \bar{a}\partial_{\bar{a}})) \,, & J_R^u &= \partial_u \,, \\
J_L^+ &= -(\partial_a + iH\bar{a}\partial_v) \,, & J_R^+ &= e^{iHu}(\partial_a - iH\bar{a}\partial_v) \,, \\
J_L^- &= -(\partial_{\bar{a}} - iHa\partial_v) \,, & J_R^- &= e^{-iHu}(\partial_{\bar{a}} + iHa\partial_v) \,.
\end{aligned}
\tag{B.11}
$$

The generators T generates translation in v, the generator $J_R - J_L$ and $J_L + J_R$ generate translation in the u direction and rotation in the transverse plane. The other generators generates some twisted translations in the transverse plane. Overall, the symmetry group is 7-dimensional and consists of two commuting copies of the Nappi-Witten algebra.

B.1 Representation theory

The representation theory of this algebra was first discussed in [9, 10], the wave functions were partially discusssed in [11, 6].

To construct the unitary irreducible representation of the Nappi-witten algebra we first identify the operators that commute with all the generators of the algebra. There are two such operators, the central generator, T, and the quadratic Casimir, $\mathcal{C}$. The former acts like a scalar in an irreducible representation:

$$T = ip^+ \,,$$

where p^+ being the light-cone momentum. The Nappi-Witten algebra admits three types of unitary representations. The unitary condition means that

$$(J^+)^\dagger = -J^- \,; \qquad T^\dagger = -T \,;\; J^\dagger = -J \,. \tag{B.12}$$

We now suppose that $p^+ > 0$. In this case we can define creation and annihilation operators

$$a = \frac{J^+}{i\sqrt{2p^+}} \,; \qquad a^\dagger = \frac{J^-}{i\sqrt{2p^+}} \,; \tag{B.13}$$

$$\left[a, a^\dagger\right] = 1 \,. \tag{B.14}$$

If we denote by $N = a^\dagger a$ the level operator, we see that $J + iN$ commutes with everybody. It is a scalar, which we denote by $-ip^-$. Once p^+, p^- are fixed the representation is uniquely determined, we denote this representation by V^{p^+,p^-}, it admits a vaccua $|0, p^+, p^-\rangle$ annihilated by $a \sim J^+$. We will denote it $|0\rangle$ for short, if there is no ambiguity. The full representation ($|n\rangle$, $n \geq 0$) is obtained by the action of $a^\dagger \sim J^-$ on the vaccua,

$$|n\rangle = (-iJ^-)^n |0\rangle \,. \tag{B.15}$$

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The symmetry generators acting on V^{p^+,p^-}, $p^+ > 0$ are given by

$$\begin{aligned} T|n\rangle &= ip^+|n\rangle\,, \\ J^+|n\rangle &= i2p^+n|n-1\rangle\,, \\ J^-|n\rangle &= i|n+1\rangle\,, \\ J|n\rangle &= i(p^- - n)|n\rangle\,. \end{aligned} \tag{B.16}$$

For our purpose it is more convenient to use the coherent basis representation The coherent states are constructed as

$$|\lambda\rangle = \exp(-\lambda J^-)|0\rangle \tag{B.17}$$

such that it is an eigenstate of the lowering operator with eigenvalue $2p^+\lambda$:

$$J^+|\lambda\rangle = 2\,p^+\lambda\,|\lambda\rangle\,. \tag{B.18}$$

The scalar product is given by

$$\langle n|m\rangle = (2p^+)^n n!\delta_{n,m} \tag{B.19}$$

and the quadratic Casimir in this representation is

$$\mathcal{C} = -2p^+\left(p^- + \frac{1}{2}\right). \tag{B.20}$$

In this representation $-iJ$ admits a highest weight p^-.

We can construct in a similar way the conjugate representation denoted $\widetilde{V}^{p^+,p^-}$. In this representation generators act as $\mathcal{P}(J), \mathcal{P}(T), \mathcal{P}(J^\pm)$ with J^+ being a creation operator and J^- an annihilation operator. $-iJ$ admits a lowest weight given by $-p^-$. The symmetry generators act as

$$\begin{aligned} T|n\rangle &= -ip^+|n\rangle\,, \\ J^+|n\rangle &= i|n+1\rangle\,, \\ J^-|n\rangle &= 2ip^+n|n-1\rangle\,, \\ J|n\rangle &= i(-p^- + n)|n\rangle\,. \end{aligned} \tag{B.21}$$

Finally the scalar product between the states is

$$\langle n|m\rangle = (2p^+)^n n!\delta_{n,m}\,. \tag{B.22}$$

The conjugate coherent basis is constructed using J^+

$$|\bar\lambda\rangle = \exp(-\bar\lambda J^+)|0\rangle\,; \tag{B.23}$$

whereas J^- is diagonal with eigenvalue $2p^+\bar\lambda$

$$J^-|\bar\lambda\rangle = 2\,p^+\bar\lambda\,|\bar\lambda\rangle\,. \tag{B.24}$$

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V^{p^+,p^-} and $\widetilde{V}^{p^+,p^-}$ share the same value of the quadratic Casimir. They are not equivalent but they are related by charge conjugation. The conjugate representation is dual to the original representation in the sense that there exists a nondegenerate invariant pairing $Q : V^{p^+,p^-} \otimes \widetilde{V}^{p^+,p^-} \to \mathbb{C}$

$$Q(|n\rangle, |m\rangle) = (-1)^n (2p^+)^n n! \delta_{n-m} \,. \tag{B.25}$$

When $p^+ = 0$ and $p^- \neq 0$, we can construct [9] a series of representations V_α^{0,p^-} labelled by p^- and a positive number α and such that a basis is given by $|n\rangle, n \in Z$. There are no highest or lowest state and the action of the symmetry generators is given by

$$\begin{aligned} T|n\rangle &= 0\,, \\ J^+|n\rangle &= i\alpha|n-1\rangle\,, \\ J^-|n\rangle &= i\alpha|n+1\rangle\,, \\ J|n\rangle &= i(p^- - n)|n\rangle\,. \end{aligned} \tag{B.26}$$

The scalar product is given by

$$\langle n|m\rangle = \delta_{n,m}, \tag{B.27}$$

and the Casimir by

$$\mathcal{C} = -\alpha^2\,. \tag{B.28}$$

These representations do not admit highest or lowest weight states. Also, are not all inequivalent, if we shift the labels n by one we get a representation where p^- is also shifted

$$V_\alpha^{0,p^-} \sim V_\alpha^{0,p^-+1}\,. \tag{B.29}$$

The last unitary representation is the trivial representation $V^{0,0}$.

B.2 Coherent states

A convenient basis to work with is the coherent state basis. From now on, we consider $p^+ > 0$, the Hilbert space V^{p^+,p^-}. It will be very convenient for us to work in the coherent state basis of V^{p^+,p^-}

$$|\lambda\rangle = \exp(-\lambda J^-)\,|0\rangle\,, \tag{B.30}$$

in which the symmetry generators are represented by differential operators

$$\begin{aligned} T|\lambda\rangle &= ip^+\,|\lambda\rangle\,, \\ J^+|\lambda\rangle &= 2p^+\lambda\,|\lambda\rangle\,, \\ J^-|\lambda\rangle &= -\partial_\lambda\,|\lambda\rangle\,, \\ J|\lambda\rangle &= i(p^- - \lambda\partial_\lambda)\,|\lambda\rangle\,. \end{aligned} \tag{B.31}$$

The conjugate state is given by

$$\langle\lambda| = \langle 0|\exp(\bar{\lambda}J^+)\,. \tag{B.32}$$

Note that here we are abusing notations, using $|\lambda\rangle$ instead of $|\lambda, p^+, p^-\rangle$. The scalar product of such states is given by

$$\langle\rho|\lambda\rangle = e^{2p^+\bar{\rho}\lambda}\,, \tag{B.33}$$

and the decomposition of unity by

$$1 = \frac{2p^+}{\pi}\int d^2\lambda\; e^{-2p^+\lambda\bar{\lambda}}\; |\lambda\rangle\langle\lambda|\,. \tag{B.34}$$

In the coherent state basis of $\widetilde{V}^{p^+,p^-}$

$$|\lambda\rangle = \exp(-\lambda J^+)|0\rangle\,, \tag{B.35}$$

The symmetry generators are given by

$$\begin{aligned} T|\lambda\rangle &= -ip^+\;|\lambda\rangle\,,\\ J^+|\lambda\rangle &= -\partial_\lambda|\lambda\rangle\,,\\ J^-|\lambda\rangle &= 2p^+\lambda\;|\lambda\rangle\,,\\ J|\lambda\rangle &= i(-p^- + \lambda\partial_\lambda)\;|\lambda\rangle\,. \end{aligned} \tag{B.36}$$

The scalar product of such states is given by

$$\langle\rho|\lambda\rangle = e^{-2p^+\bar{\rho}\lambda}\,. \tag{B.37}$$

When $p^+ = 0$ we define the 'coherent state' of the representation V_α^{0,p^-} to be

$$|\theta\rangle = \sum_{n\in\mathbb{Z}} e^{in\theta}|n\rangle\,. \tag{B.38}$$

The symmetry generators are given by

$$\begin{aligned} T|\theta\rangle &= -ip^+\;|\theta\rangle\,,\\ J^+|\theta\rangle &= i\alpha e^{i\theta}|\theta\rangle\,,\\ J^-|\theta\rangle &= i\alpha e^{-i\theta}|\theta\rangle\,,\\ J|\theta\rangle &= i(p^- - \partial_\theta)\;|\lambda\rangle\,, \end{aligned} \tag{B.39}$$

and the scalar product by

$$\langle\phi|\theta\rangle = \delta(\phi-\theta)\,. \tag{B.40}$$

B.3 Wave functional

As we have already seen a Nappi-Witten group element can be parametrized by the coordinates $a, \bar{a}, u, v$

$$g(a,u,v) = e^{HaJ^+ + H\bar{a}J^-}e^{HuJ + HvT}\,, \tag{B.41}$$

where $a = a_1 + i\,a_2$. We want to compute the matrix element of g in the coherent state basis of V^{p^+,p^-}, let us denote

$$\phi^{p^+,p^-}_{\bar{\rho},\lambda}(g) = \langle\rho|g|\lambda\rangle\,. \tag{B.42}$$

With the help of the formulae (B.30), (B.4) and after some algebra where one pushes the J^+ to the right and the J^- to the left, one gets

$$\phi^{p^+,p^-}_{\bar{\rho},\lambda}(g) = e^{ip^+v+ip^-u}e^{-p^+a\bar{a}}\exp\left[2p^+(a\lambda e^{-iu}-\bar{\rho}\bar{a})+2p^+\bar{\rho}\lambda e^{-iu}\right]. \tag{B.43}$$

The ground state wave functional is

$$\phi^{p^+,p^-}_{0,0}(g) = e^{ip^+v+ip^-u}e^{-p^+a\bar{a}}. \tag{B.44}$$

This corresponds to a plane wave centered in the transverse plane around $a = 0$. In general this wave functional can be written in a more suggestive form as

$$\begin{aligned}\phi^{p^+,p^-}_{\bar{\rho},\lambda}(a,u,v) = {} & e^{ip^+v+ip^-u}\times \\ & \times e^{-p^+|\bar{a}-\lambda e^{-iu}+\rho|^2}e^{p^+a(\rho+\lambda e^{-iu})-\text{c.c}}\times \\ & \times e^{p^+|\lambda|^2+p^+|\rho|^2-p^+(\bar{\lambda}\rho e^{iu}-\lambda\bar{\rho}e^{-iu})}.\end{aligned} \tag{B.45}$$

This expression clearly shows that the wave functional is centered around $\bar{a} = -\rho+\lambda e^{-iu}$. Moreover, it is a plane wave in v of momentum p^+ and the semi-classical momentum p^a in the transverse plane is given by $\rho+\lambda e^{iu}$. The direction u has a more complicated semi-classical momentum given by $p^u = p^- - \lambda(a+\bar{\rho})e^{-iu} + \text{c.c.}$ All these results agree with the analysis of the geodesics in Nappi-Witten space.

Using the same techniques we can construct the matrix elements of g in the conjugate representation

$$\widetilde{\phi}^{p^+,p^-}_{\bar{\rho},\lambda}(g) \equiv \phi^{p^+,p^-}_{\bar{\rho},\lambda}(\mathcal{P}(g)) = e^{-ip^+v-ip^-u}e^{-p^+a\bar{a}}\exp\left[2p^+(\bar{a}\lambda e^{+iu}-\bar{\rho}a)+2p^+\bar{\rho}\lambda e^{+iu}\right]. \tag{B.46}$$

This corresponds to a wave moving in backwards in light-cone time and centered around $a = \lambda e^{iu}-\rho$. The conjugate wave functional is related to the original one by complex conjugation

$$\widetilde{\phi}^{p^+,p^-}_{\bar{\rho},\lambda}(g) = \overline{\phi^{p^+,p^-}_{\rho,\bar{\lambda}}(g)}. \tag{B.47}$$

One can finally construct the wave functional associated with the $p^+ = 0$ representations. The matrix elements of g in the representation V^{0,p^-}_α are given by

$$\langle\phi|g|\theta\rangle = \delta(\phi-\theta+u)e^{ip^-u}e^{i\alpha e^{i\theta}e^{-iu}a+i\alpha e^{-i\theta}e^{iu}\bar{a}}. \tag{B.48}$$

B.4 Wave equation

The wave functions $\phi^{p^+,p^-}_{\bar{\rho},\lambda}$ are a complete basis of solutions of the wave equations

$$\partial_v\phi = ip^+\phi, \qquad \Delta\phi = -2p^+\left(p^- - \frac{1}{2}\right)\phi, \tag{B.49}$$

where $p^+ > 0$ and

$$\Delta = 2\partial_u\partial_v + \partial_a\partial_{\bar{a}} + a\bar{a}\partial_v\partial_v + i(a\partial_a - \bar{a}\partial_{\bar{a}})\partial_v, \tag{B.50}$$

is the laplacian of the pp-wave metric. This laplacian is also equal to the Casimir for the left or right symmetry generators (B.11)

$$\begin{aligned}\Delta &= 2J_LT_L + \frac{(J_L^+J_L^- + J_L^-J_L^+)}{2}\,,\\ &= 2J_RT_R + \frac{(J_R^+J_R^- + J_R^-J_R^+)}{2}\,.\end{aligned} \tag{B.51}$$

This can be checked directly but can also be understood by the fact that the wave function $\phi^{p^+,p^-}_{\bar\rho,\lambda}$ intertwines the action of the left symmetry generators with the coherent state generators (B.31) acting on $\bar\rho$. One can check that $\phi^{p^+,p^-}_{\bar\rho,\lambda}(g)$ is a solution of

$$\begin{aligned}T_L\phi(g) &= -ip^+\phi(g)\,, & T_R\phi(g) &= ip^+\phi(g)\,,\\ J_L^+\phi(g) &= -\partial_{\bar\rho}\phi(g)\,, & J_R^+\phi(g) &= 2p^+\lambda\,\phi(g)\\ J_L^-\phi(g) &= 2p^+\bar\rho\,\phi(g)\,, & J_R^-\phi(g) &= -\partial_\lambda\phi(g)\\ J_L\phi(g) &= -i(p^- - \bar\rho\partial_{\bar\rho})\phi(g)\,, & J_R\phi(g) &= i(p^- - \lambda\partial_\lambda)\phi(g)\,.\end{aligned} \tag{B.52}$$

This is clear by the construction of the wave function since

$$J_R^+\langle\rho|g|\lambda\rangle = \langle\rho|gJ^+|\lambda\rangle = 2p^+\lambda\langle\rho|g|\lambda\rangle, \tag{B.53}$$

and similarly for the other generators.

The wave functions $\widetilde\phi^{p^+,p^-}_{\bar\rho,\lambda}$ are solutions of the wave equations

$$\partial_v\phi = -ip^+\phi\,; \qquad \Delta\phi = -2p^+\left(p^- + \frac{1}{2}\right)\phi\,. \tag{B.54}$$

The action of the symmetry generators on $\widetilde\phi^{p^+,p^-}_{\bar\rho,\lambda}$ is obtained from (B.52) by exchanging the left symmetry generators with the right generators.

It is important for us to note that $\phi^{-p^+,-(p^-+1)}_{\bar\rho,\lambda}(g)$ is also solution of eq. (B.49). This solution is well defined as a function on the group, however it contains a factor $\exp(2p^+a\bar a)$ which is unbounded. It is then not normalisable and can not be taken as a wave functional corresponding to the representation $\widetilde V^{p^+,p^-}$. However one can check that the integral transform of this solution

$$\widehat\phi^{p^+,p^-}_{\bar\rho,\rho}(g) \equiv \int e^{2p^+\lambda\rho}e^{2p^+\bar\lambda\bar\rho}\,\phi^{-p^+,-(p^-+1)}_{\bar\lambda\lambda}(g)\,d\lambda d\bar\lambda\,, \tag{B.55}$$

satisfies the same transformation rules as $\widetilde\phi^{p^+,p^-}_{\bar\rho,\rho}$ under the symmetry transformation. In order to show this we only need to use rule of integration by part

$$\widehat\phi^{p^+,p^-}_{\rho,\lambda}(g) = \widehat\phi^{p^+,p^-}_{\rho,\lambda}(1)\widetilde\phi^{p^+,p^-}_{\rho,\lambda}(g)\,. \tag{B.56}$$

This relation is formal[20] if we do not check the convergence properties of the integral transform. Ideally we would have to specify a proper choice of contour of integration in order for this relation to be rigourously valid. In appendix C we give the rules for computing such integral transforms.

[20]The proportionality coefficient between $\widehat\phi$ and $\widetilde\phi$ could be infinite.

B.5 Plancherel formula

We denote by $\chi^{p^+,p^-}(g) = \mathrm{tr}_{V^{p^+,p^-}}(g)$ the character of the representation V^{p^+,p^-}, $\widetilde{\chi}^{p^+,p^-}(g)$ and $\chi^{0,p^+}_{\alpha}(g)$ the one associated with $\widetilde{V}^{p^+,p^-}$ and V^{0,p^-}_{α}. A direct computation using $\chi^{p^+,p^-}(g) = \frac{2p^+}{\pi}\int e^{-2p^+\lambda\bar{\lambda}}\,\phi^{p^+,p^-}_{\bar{\lambda}\lambda}(g)d^2\lambda$ shows that

$$\chi^{p^+,p^-}(g) = \frac{e^{ip^+v}e^{ip^-u}}{1-e^{-iu}}\exp\left(ip^+\frac{\cos u/2}{\sin u/2}a\bar{a}\right) \tag{B.57}$$

when $u \neq 0(2\pi)$. If $u = 0(2\pi)$ then $\chi^{p^+,p^-}(g) = \frac{\pi}{2p^+}\delta^2(a)$. Also, $\widetilde{\chi}^{p^+,p^-}(g) = \overline{\chi^{p^+,p^-}(g)}$ and

$$\chi^{0,p^-}_{\alpha}(g) = \delta(u)J_0(2\alpha|a|)\,, \tag{B.58}$$

where J_0 is the Bessel function. Now

$$\int_{-\infty}^{+\infty} dp^-\chi^{p^+,p^-}(g) = \frac{\pi^2}{p^+}e^{ip^+v}\delta(u)\delta^2(a)\,. \tag{B.59}$$

We therefore have the Plancherel formula

$$\frac{1}{\pi^2}\int_0^{+\infty} p^+dp^+\int_{-\infty}^{+\infty} dp^-(\chi^{p^+,p^-}(g) + \widetilde{\chi}^{p^+,p^-}(g)) = \delta(g)\,. \tag{B.60}$$

B.6 More representation theory

The way the product of two waves functions decomposes as a linear sum of wave function gives us a lot of important information from the physical and mathematical side. From the mathematical side it contains all the information we need on the tensor product of representation and the recoupling coefficient involved in the tensorisation. From the physical side we can read out what are the conservation rules and how two wave interact in our curved background. We are interested in the multiplicative properties of the wave functions

$$\begin{aligned}\phi^{p^+,p^-}_{\bar{\rho},\lambda}(g) &= e^{ip^+(v+ia\bar{a})+ip^-u}e^{2p^+a\lambda e^{-iu}}e^{-2p^+\bar{\rho}\bar{a}}e^{2p^+\bar{\rho}\lambda e^{-iu}}\\ \widetilde{\phi}^{p^+,p^-}_{\bar{\rho},\lambda}(g) &= e^{-ip^+(v-ia\bar{a})-ip^-u}e^{2p^+\bar{a}\lambda e^{+iu}}e^{-2p^+\bar{\rho}a}e^{2p^+\bar{\rho}\lambda e^{+iu}}\,.\end{aligned} \tag{B.61}$$

It is easy to see that if we define $p_3^{\pm} = p_1^{\pm} + p_2^{\pm}$, $p_3^+\lambda_3 = p_1^+\lambda_1 + p_2^+\lambda_2$, $p_3^+\rho_3 = p_1^+\rho_1 + p_2^+\rho_2$ the product is given by

$$\phi^{p_1^+,p_1^-}_{\bar{\rho}_1,\lambda_1}(g)\phi^{p_2^+,p_2^-}_{\bar{\rho}_2,\lambda_2}(g) = \phi^{p_3^+,p_3^-}_{\bar{\rho}_3,\lambda_3}(g)\exp e^{-iu}(2p_1^+\bar{\rho}_1\lambda_1 + 2p_2^+\bar{\rho}_2\lambda_2 - 2p_3^+\bar{\rho}_3\lambda_3)\,. \tag{B.62}$$

The term in the exponent can be evaluated to be

$$\frac{p_1^+p_2^+}{p_3^+}(\lambda_1-\lambda_2)(\bar{\rho}_1-\bar{\rho}_2)e^{-iu}\,. \tag{B.63}$$

If we Taylor expand the exponential and utilize the fact that $e^{-iu}\phi^{p^+,p^-}_{\bar{\rho},\lambda} = \phi^{p^+,p^--1}_{\bar{\rho},\lambda}$, we get the simple result

$$\phi^{p_1^+,p_1^-}_{\bar{\rho}_1,\lambda_1}(g)\phi^{p_2^+,p_2^-}_{\bar{\rho}_2,\lambda_2}(g) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{p_1^+p_2^+}{p_3^+}\right)^n(\lambda_1-\lambda_2)^n(\bar{\rho}_1-\bar{\rho}_2)^n\phi^{p_3^+,p_3^--n}_{\bar{\rho}_3,\lambda_3}(g)\,. \tag{B.64}$$

From which we read out the tensorisation rules

$$V^{p_1^+,p_1^-}\otimes V^{p_2^+,p_2^-}=\sum_{n=0}^{+\infty}V^{p_1^++p_2^+,p_1^-+p_2^--n}\,, \tag{B.65}$$

and the Clebsh-Gordan coefficients

$$C^{p_1^\pm p_2^\pm\lambda_3}_{\lambda_1\lambda_2p_3^\pm}=\sum_n\delta(p_1^++p_2^+-p_3^+)\delta(p_1^-+p_2^--p_3^--n)\delta^2(\lambda_1+\lambda_2-\lambda_3)\frac{1}{\sqrt{n!}}\left(\frac{p_1^+p_2^+}{p_3^+}\right)^{n/2}(\lambda_1-\lambda_2)^n\,. \tag{B.66}$$

The product of two conjugate wave function is similar since $\widetilde{\phi}(g)=\phi(\mathcal{P}(g))$ and so is the tensorisation rule and the Clebsch-Gordan coefficients for the conjugate representations $\widetilde{V}$.

C. Integral transform

We define the integral transform to be the following linear transformation

$$I(\lambda^s\bar\lambda^{\bar s})(\rho,\bar\rho)=\frac{\Gamma(\bar s+1)}{\Gamma(-s)}(-\bar\rho)^{-\bar s-1}(\rho)^{-s-1}\,, \tag{C.1}$$

with $s-\bar s$ restricted to be an integer. This condition this expression is symmetric in the exchange of s with $\bar s$. One can check that it satisfies the following properties

$$\begin{aligned}I(e^{\bar a\bar\lambda}(\lambda\bar\lambda)^s)(\rho,\bar\rho)&=I((\lambda\bar\lambda)^s)(\rho,\bar\rho+\bar a)\\ I((\lambda+a)^s\bar\lambda^{\bar s})(\rho)&=e^{-\rho a}I(\lambda^s\bar\lambda^{\bar s})(\rho)\\ I\circ I((\lambda^s\bar\lambda^{\bar s})(\rho)&=-(-1)^{s-\bar s}\rho^s\bar\rho^{\bar s}\end{aligned} \tag{C.2}$$

the first two identities express the fact that I represent the integral transform

$$I(f)(\rho,\bar\rho)=-\frac{1}{\pi}\int d^2\lambda|e^{\rho\lambda}|^2f(\lambda,\bar\lambda)\,. \tag{C.3}$$

where the normalization has been fixed by computing the integral transform of $e^{\lambda\bar\lambda}$. In order to prove these equalities we need to use the following identities

$$\begin{aligned}e^{-z}&=\sum_n\frac{(-1)^n}{\Gamma(n+1)}z^n\,,\\ (z+w)^s&=\sum_n\frac{(-1)^n\Gamma(n-s)}{\Gamma(n+1)\Gamma(-s)}z^nw^{n-x}\end{aligned} \tag{C.4}$$

when $z<w$. Let us prove for instance the first identity (C.2)

$$\begin{aligned}I(e^{\bar a\bar\lambda}(\lambda\bar\lambda)^s)(\rho,\bar\rho)&=\sum_n\frac{\bar a^n}{\Gamma(n+1)}I(\lambda^n\bar\lambda^{n+s})\\ &=\sum_n\frac{\Gamma(n+s+1)}{\Gamma(n+1)\Gamma(-s)}\bar a^n\rho^{-s-1}(-\bar\rho)^{-n-s-1}\\ &=\frac{\Gamma(s+1)}{\Gamma(-s)}(-\bar\rho\rho)^{-s-1}\sum_n\frac{\Gamma(n+s+1)}{\Gamma(n+1)\Gamma(s+1)}\left(-\frac{\bar a}{\bar\rho}\right)^n\\ &=\frac{\Gamma(s+1)}{\Gamma(-s)}(-\rho(\bar\rho+\bar a))^{-s-1}\end{aligned} \tag{C.5}$$

which is the r.h.s. of (C.2). The main result of this section consists in showing the following identity

$$I\left(\frac{(A\bar{\lambda}\lambda+\bar{B}\lambda+B\bar{\lambda}+C)^s}{\Gamma(s+1)}\right)(\rho,\bar{\rho}) = -\left|e^{-\rho\frac{B}{A}}\right|^2 \frac{A^s}{(-|\rho|^2)^{s+1}}\left(\frac{\mathcal{D}}{2}\right)^{s+1} I_{-s-1}(\mathcal{D})\,, \quad \text{(C.6)}$$

where we have introduced

$$\frac{\mathcal{D}}{2} = |\rho|\frac{\sqrt{B\bar{B}-AC}}{A}\,, \quad \text{(C.7)}$$

and $I_\nu(z)$ is the modified Bessel function of the first kind.

Here is the proof: first we complete the square

$$(A\bar{\lambda}\lambda+\bar{B}\lambda+B\bar{\lambda}+C) = A\left(\left|\lambda+\frac{B}{A}\right|^2+\frac{AC-B\bar{B}}{A^2}\right) \quad \text{(C.8)}$$

then use the identity (C.4) to express the l.h.s. of (C.6) as

$$\sum_n \frac{(-1)^n\Gamma(n-s)}{\Gamma(n+1)\Gamma(-s)\Gamma(s+1)} A^s I\left(\left|\lambda+\frac{B}{A}\right|^{2(s-n)}\right)(\rho)\left(\frac{AC-B\bar{B}}{A^2}\right)^n\,, \quad \text{(C.9)}$$

after evaluation of the integral transform one get

$$\left|e^{-\rho\frac{B}{A}}\right|^2 \frac{A^s}{(-|\rho|^2)^{s+1}}\frac{\sin\pi(s+1)}{\pi}\sum_n \frac{(-1)^n\Gamma(s+1-n)}{\Gamma(n+1)}\left(-|\rho|^2\frac{AC-B\bar{B}}{A^2}\right)^n\,. \quad \text{(C.10)}$$

To finish the proof one need to use the identity

$$\sum_n \frac{(-1)^n\Gamma(\nu-n)}{\Gamma(n+1)}\left(\frac{z}{2}\right)^{2n} = -\frac{\pi}{\sin\pi\nu}\left(\frac{z}{2}\right)^\nu I_{-\nu}(z)\,. \quad \text{(C.11)}$$

D. Chiral splitting of integrals

In this section we prove the chiral splitting property of the integral

$$I(f,\bar{g}) = \int d^2w\, f(w)\,\bar{g}(\bar{w})\,, \quad \text{(D.1)}$$

where $f,\bar{g}$ are analytic functions having branch points at $z_1,\dots,z_n$. The monodromy around the point z_i is given by $e^{2i\pi p_i}$. For example we can have $f(w)=\prod_i(w-z_i)^{p_i}$. The main point we want to stress here is that such surface integrals can be written in terms of contour integrals. Suppose for simplicity that all z_i are real and ordered as $z_1<z_2<\cdots<z_n$. We then have

$$\int d^2w\; f(w)\,\bar{g}(w) = \sum_{i=1}^{n-1}\oint_{a_i} f(w)\,dw\int_{z_i}^{z_{i+1}}\bar{g}(\bar{w})d\bar{w}\,, \quad \text{(D.2)}$$

where a_i is the contour of integration starting at $-\infty$ going around $z_1,\dots,z_i$ in a clockwise direction and going back to $-\infty$ below the real axis. The proof of this statement is obtained by writing the integral over $w=x+iy$ as an integral over x and y and then rotating the contour of integration of iy along the real axis. A careful analysis similar to the one done in [42] leads to the results stated above.

We can now specify this result when there are three finite branch points $0, z, 1$ with $0 < z < 1$. Let us introduce the following notation $I_1(f) = \int_{-\infty}^0 |f(w)|dw$, $I_2(f) = \int_0^z |f(w)|dw$, $I_3(f) = \int_z^1 |f(w)|dw$, $I_4(f) = \int_1^\infty |f(w)|dw$. These integrals are not all independent. Consider the contour of integration C starting from $+\infty$ going around $0, z, 1$ in a clockwise manner and going back to ∞. The integral of f along this contour is zero; and it can be decomposed as integrals I_i. Namely

$$0 = \oint_C f = \sin \pi p_1 I_2 + \sin \pi(p_1 + p_2) I_3 + \sin \pi(p_1 + p_2 + p_3) I_4 \,. \tag{D.3}$$

If we now take a contour which starts at $-\infty$, goes around $0, z$ in a clockwise direction, comes back at $-\infty$ and then another contour which starts off at $+\infty$ and goes around 1 in a counterclockwise direction, we get another identity

$$0 = \sin \pi(p_1 + p_2) I_1 + \sin \pi p_2 I_2 - \sin \pi p_3 I_4 \,. \tag{D.4}$$

The integrals along the contours a_1, a_2 can also be expressed in terms of I_i

$$\begin{aligned} \oint_{a_1} \frac{d^2\lambda}{\pi} f &= -\sin \pi p_1 I_1(f) \\ \oint_{a_2} \frac{d^2\lambda}{\pi} f &= -\sin \pi p_3 I_4(f) \,. \end{aligned} \tag{D.5}$$

The measure in the above integrals are fixed by computing the integral transform of $e^{\lambda\bar{\lambda}}$. Altogether the total integral can be written as

$$-\sin \pi p_1 I_1(f) \bar{I}_2(g) - \sin \pi p_3 I_4(f) \bar{I}_3(g) \,, \tag{D.6}$$

Using eq. (D.3) and eq. (D.4) we can eliminate $I_1(f)$ and $\bar{I}_3(g)$ respectively. This finally gives the identity

$$I(f, \bar{g}) = \frac{\sin \pi p_1 \sin \pi p_2}{\sin \pi(p_1 + p_2)} I_2(f) \bar{I}_2(g) + \frac{\sin \pi p_3 \sin \pi(p_1 + p_2 + p_3)}{\sin \pi(p_1 + p_2)} I_4(f) \bar{I}_4(g) \,. \tag{D.7}$$

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